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JULIA SETS AND BIFURCATION DIAGRAMS FOR EXPONENTIAL MAPS

BY ROBERT L. DEVANEY

ABSTRACT. We describe some of the bifurcations that occur in the family of entire maps $E_\lambda(z) = \lambda \exp(z)$. When $\lambda = 1$, it is known that $J(E_\lambda) = \mathbf{C}$. We show that there are many other values for which this happens. However, in each case, there are nearby λ-values for which $J(E_\lambda)$ is nowhere dense.

Let $F(z)$ be an entire transcendental function. The Julia set of F, denoted by $J(F)$, is the set of points at which the family of iterates of F (i.e. F, $F \circ F = F^2$, $F^3, \ldots$) fails to be a normal family. Equivalently, $J(F)$ is the closure of the set of nonattracting periodic points of F. It is known **[F]** that $J(F)$ is a closed, nonempty, perfect set which is invariant under F and all branches of F^{-1}. Moreover, most of the interesting chaotic dynamics of the map occur on the Julia set.

There has been much recent work on the structure of the Julia set of complex analytic functions **[B, DH, R, S, MSS]**. Most of this work is restricted to the polynomial or rational function case. Our goal in this note is to point out that, while entire functions share many of the properties of these maps, there are several significant differences.

We will concentrate on the one-parameter family of maps $E_\lambda(z) = \lambda \exp(z)$ where $\lambda \in \mathbf{C}$. Similar results hold for other classes of entire maps, e.g. $z \to a + b \sin(z)$ and $z \to Q(z) \exp(P(z))$ where P and Q are polynomials. A major difference between the exponential family and polynomials is the possibility that $J(E_\lambda) = \mathbf{C}$. Indeed, Misiurewicz **[M]** has recently shown that $J(e^z) = \mathbf{C}$, answering affirmatively a sixty year old question of Fatou **[F]**. It turns out that this is a common occurrence for entire maps.

1. The exponential family. The family of functions E_λ is a natural family of complex analytic maps in the sense that any entire function which is topologically conjugate to some E_λ is in fact affinely equivalent to a member of this family. This essentially follows from the fact that E_λ has no critical points and one omitted value. The orbit of the omitted value at 0 is the crucial factor which governs the dynamics of E_λ.

PROPOSITION. 1. *If $E_\lambda^n(0) \to \infty$, then $J(E_\lambda) = \mathbf{C}$.*

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