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THE NEWTON DIAGRAM OF AN ANALYTIC MORPHISM, AND APPLICATIONS TO DIFFERENTIABLE FUNCTIONS

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Consider a system of equations of the form

$$f(x) = A(x) \cdot g(\phi(x)), \tag{1}$$

where $x = (x_1, \ldots, x_m)$, $\phi(x) = (\phi_1(x), \ldots, \phi_n(x))$ is an analytic mapping, and $A(x)$ is a $p \times q$ matrix of analytic functions. Given $f(x) = (f_1(x), \ldots, f_p(x)) C^\infty$, we seek C^∞ solutions $g(y) = (g_1(y), \ldots, g_q(y))$. There is a necessary condition on the Taylor series of f at each point. Special cases are classical: when $\phi(x) \equiv x$ we have the division theorem of Malgrange [**7**, Chapter VI], and when $A(x) \equiv I$, the composition problem first studied by Glaeser [**5**].

We solve the problem in the case that $\phi(x)$ and $A(x)$ are algebraic (or Nash), using a Hilbert-Samuel stratification associated to (1). Our methods, however, go far beyond this case. We present algebraic criteria for solving (1), based on a fundamental relationship between two invariants of an analytic morphism and an associated "Newton diagram". Hironaka's simple but powerful formal division algorithm [**3**] is exploited systematically. The only results from "differential analysis" used are Whitney's extension theorem [**7**, Chapter I] and Łojasiewicz's inequality [**7**, Chapter IV].

Let $\underline{k} = \mathbf{R}$ or $\mathbf{C}$. (Some of our assertions hold for other fields.) Let M, N be analytic manifolds (over $\underline{k}$), and $\phi\colon M \to N$ an analytic mapping. Let A be a $p \times q$ matrix of analytic functions on M.

For each $a \in M$, let $\mathcal{O}_a$ (respectively, $\hat{\mathcal{O}}_a$) denote the ring of germs of analytic functions at a (respectively, the completion of $\mathcal{O}_a$ in the Krull topology). Let $\hat{\mathfrak{m}}_a$ be the maximal ideal of $\hat{\mathcal{O}}_a$. In the case $\underline{k} = \mathbf{R}$, let $C^\infty(M)$ denote the algebra of C^∞ functions on M. There is a Taylor series homomorphism $f \mapsto \hat{f}_a$ from $C^\infty(M)^p$ onto $\hat{\mathcal{O}}_a^p$.

The mapping ϕ induces ring homomorphisms $\phi^*\colon C^\infty(N) \to C^\infty(M)$, $\phi_a^*\colon \mathcal{O}_{\phi(a)} \to \mathcal{O}_a$, and $\hat{\phi}_a^*\colon \hat{\mathcal{O}}_{\phi(a)} \to \hat{\mathcal{O}}_a$. Let $\Phi\colon C^\infty(N)^q \to C^\infty(M)^p$ denote the module homomorphism over ϕ^* defined by $\Phi(g) = A \cdot (g \circ \phi)$. Let $\hat{\Phi}_a\colon \hat{\mathcal{O}}_{\phi(a)}^q \to \hat{\mathcal{O}}_a^p$ denote the analogous module homomorphism over $\hat{\phi}_a^*$.

Let $(\Phi C^\infty(N)^q)\hat{}$ denote the $C^\infty(N)$-submodule of $C^\infty(M)^p$ consisting of elements which formally belong to the image $\Phi C^\infty(N)^q$ of Φ; i.e., $(\Phi C^\infty(N)^q)\hat{} = \{f \in C^\infty(M)^p\colon$ for all $b \in \phi(M)$, there exists $G_b \in \hat{\mathcal{O}}_b$ such that $\hat{f}_a = \hat{\Phi}_a(G_b)$ for all $a \in \phi^{-1}(b)\}$. Evidently, $(\Phi C^\infty(N)^q)\hat{}$ is closed in the C^∞ topology.

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