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SOME PROBLEMS IN POTENTIAL THEORY AND THE NOTION OF HARMONIC ENTROPY

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ABSTRACT. Blaschke regions are studied for certain classes of subharmonic functions in connection with the notion of harmonic entropy. A complete description of Riesz measures for some of these classes is obtained. A new analytic inequality is established.

1. Definitions, notations and two basic problems. $k(r)$ $(0 \le r < 1)$ will always denote a continuous nonnegative function such that $k(|z|)$ is subharmonic in the open unit disc $\mathbf{D}$ (or, equivalently, such that $k(r)$ and $rk'(r)$ are nondecreasing).

DEFINITION 1. Let $\mathcal{M} \subset \mathbf{D}$ be a given set, and let $\mathcal{H}_{\langle k\rangle}(\mathcal{M})$ be the set of all nonnegative harmonic functions $u(z)$ in $\mathbf{D}$ such that $u(z) \ge k(|z|)$ on $\mathcal{M}$. The following quantity will be called the *harmonic k-entropy* of $\mathcal{M}$:

$$\mathcal{E}(\mathcal{M}; k) = \min\{u(0)\colon u \in \mathcal{H}_{\langle k\rangle}(\mathcal{M})\}.^2 \tag{1.1}$$

If $\mathcal{H}_{\langle k\rangle}(\mathcal{M})$ is empty, we set $\mathcal{E}(\mathcal{M}; k) = +\infty$.

DEFINITION 2. $\mathcal{SH}^{\langle k\rangle}$ will denote the class of subharmonic functions $u(z)$ in $\mathbf{D}$ such that

$$u(z) \le C_u k(|z|) \qquad (z \in \mathbf{D}), \tag{1.2}$$

where C_u is some constant (depending on u).

DEFINITION 3. $\mathcal{A}^{\langle k\rangle}$ will denote the class of analytic functions $f(z)$ in $\mathbf{D}$ such that $\log|f(z)| \in \mathcal{SH}^{\langle k\rangle}$.

DEFINITION 4. A region $G \subset \mathbf{D}$ is called a k-Blaschke region if either of two equivalent[3] conditions holds:

(a) for every $u \in \mathcal{SH}^{\langle k\rangle}$

$$b(G; d\mu) = \int_G (1 - |z|)\, d\mu(z) < \infty, \tag{1.3}$$

where $d\mu = \Delta u$ is the Riesz measure (i.e. generalized Laplacian) of u;

(b) for every $f \in \mathcal{A}^{\langle k\rangle}$

$$\sum_{z_\nu \in G} (1 - |z_\nu|) < \infty, \tag{1.3'}$$

where $\{z_\nu\}$ is the zero set of f.

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[2]The use of that term, borrowed from Information Theory, is suggested by this interpretation: if $u(z)$ is conceived as a "signal" of strength $u(0)$ and $k(|z|)$ as the "noise", then $\mathcal{E}(\mathcal{M}; k)$ is the strength of the weakest signal that overcomes the noise on $\mathcal{M}$.

[3]The equivalence of (a) and (b) is easily proved.