BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 2, March 1983

0273-0979/82/0000-0977/$01.75

Maximum principles and their applications, by René Sperb, Mathematics in Science and Engineering, vol. 157, Academic Press, New York, 1981, ix + 224 pp., $29.50.

Maximum principles are among the most useful and best known tools in the study of second order elliptic equations. They generalize the elementary fact that a function $f(x)$ with $f''(x) > 0$ in $[a, b]$ assumes its maximum either in a or in b. The basic principles which are usually referred to are *Hopf's first* and *second principles*. The first principle states that a function u defined in a bounded domain $D \subset \mathbf{R}^N$ and satisfying the differential inequality $Lu \geqslant 0$, where

$$L = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial}{\partial x_i},$$

$x = (x_1, x_2, \ldots, x_N)$, is a uniformly elliptic operator with bounded coefficients, cannot attain its maximum in D unless it is a constant. If $u \not\equiv$ constant assumes its maximum at some point $P \in \partial D$, then it is clear that the outer normal derivative $\partial u/\partial n$ of u at this point is nonnegative. However by Hopf's second maximum principle we have even the sharper estimate $\partial u/\partial n > 0$.

By means of these results it is easy to derive uniqueness theorems for Dirichlet problems and bounds for their solutions in terms of the data. Hopf's maximum principles have been generalized in numerous ways and applied to a wide range of problems both of mathematical and physical interest. Moreover they provide useful tools in the approximation of solutions and in the determination of error bounds for such approximations. An excellent reference is Protter and Weinberger's book **[1]** which at the same time serves students and researchers in this field.

Why a new text on this topic? First of all great progress has been made in the last years and methods were developed which are rooted in the maximum principles, especially in connexion with nonlinear problems. An important method of this type is the method of upper and lower solutions **[2]**. In addition, maximum principles led to nonexistence results for certain Dirichlet problems **[3]**, and they served also to prove symmetry and convexity in elliptic and parabolic equations **[4–7]**.