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Some applications of topological K-theory, by N. Mahammed, R. Piccinini and U. Suter, Mathematics Studies, no. 45, North-Holland, Amsterdam, ix + 137 pp., $41.50.

1. *K*-theory. Topological K-theory was developed by Atiyah and Hirzebruch **[4]** by analogy with Grothendieck's K-groups in algebraic geometry. For a compact, Hausdorff space X one considers vector bundles, $\pi\colon E \to X$. That is, E is locally a product of a neighborhood U in X with a (complex, let us say) vector space so that π is locally a projection onto U. Take isomorphism classes of such (complex) vector bundles and form the group generated by these subject to the condition that the fibre-by-fibre direct sum $E \oplus F$ shall represent the group-sum of the classes of E and F. The resulting group is $K(X)$.

$K(X)$ has been called "the linear algebra of algebraic topology". As we have seen with direct sums, natural vector space operations can be performed fibre-by-fibre on vector bundles. For example, tensor product gives $K(X)$ a ring structure while the exterior power constructions, $\lambda^i E$, induce natural operations on $K(X)$ from which further useful operations can be constructed —the most celebrated of these are the Adams operations, ψ^k, which are ring homomorphisms. A map $f\colon Y \to X$ enables us to construct a vector bundle f^*E over Y, given by $f^*E = \{(e, y) \in E \times Y \mid \pi(e) = f(y)\}$. In fact $f^*\colon K(X) \to K(Y)$ is a ring homomorphism and $\psi^k f^*(z) = f^*(\psi^k(z))$—a simple property from which many famous results in algebraic topology have been deduced.

One may construct a homomorphism $\varepsilon\colon K(X) \to Z$ by choosing $x_0 \in X$ and assigning to the class of E the dimension of $\pi^{-1}(x_0)$. This homomorphism is split, by sending n to $X \times \mathbf{C}^n$, and we may set $\ker \varepsilon = \tilde{K}(X)$. Bott **[6]** proved that

$$\tilde{K}(S^n) = \begin{cases} 0 & \text{if } n \text{ odd}, \\ Z & \text{if } n \text{ even}, \end{cases}$$

thereby opening the way to calculations of $K(X)$ and thereafter to applications to algebraic topology. For example, let P^n denote the projection space obtained from the n-sphere, S^n, by identifying antipodal points. $\tilde{K}(P^n)$ provides a fine example of the "linear algebra" aspects of K-theory. A vector bundle, the Hopf bundle, H, is formed by taking $S^n \times \mathbf{C}$ and identifying (z, v) with $(-z, -v)$. In fact $H - 1$ generates $\tilde{K}(P^n)$, which is a cyclic group of 2-primary order. If one has a nondegenerate bilinear map $\phi\colon \mathbf{R}^{n+1} \times \mathbf{C}^m \to \mathbf{C}^m$ (for example: R^m might be a module over the Clifford algebra $C(\mathbf{R}^{n+1})$) then $0 = m(H - 1) \in \tilde{K}(P^n)$ by virtue of the isomorphism $S^n \times \mathbf{C}^m \to S^n \times \mathbf{C}^m$ which sends (z, v) to $(z, \phi(z, v))$ and induces an isomorphism $mH \cong P^n \times \mathbf{C}^m$. A Clifford module structure on $\mathbf{R}^m$, such as ϕ above, can be used to transport, in a continuous manner, a copy of $\mathbf{R}^{n+1}$ tangent to a chosen point on S^{m-1} in such a way as to give a subvector bundle of the form $S^{m-1} \times \mathbf{R}^{n+1}$ of the tangent vector bundle of S^{m-1}, $\tau_{S^{m-1}}$. Adams **[1]** showed that this construction