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Introduction to algebraic K-theory, by John R. Silvester, Chapman & Hall, London, 1981, xi + 255 pp., $29.95 hardcover, $13.95 paperback.

K-theory, both algebraic and topological, had its beginnings in the 1940s when Grothendieck defined a group, now known as the Grothendieck group, to use in his proof of the Riemann-Roch theorem. Algebraically, the Grothendieck group of a ring R is the abelian group generated by the projective R modules P with the relation $[P] = [P'] + [P'']$ whenever $0 \to P' \to P \to P'' \to 0$ is exact. For example, if R is a field, a local ring or a principal ideal domain, then the Grothendieck group of R is $\mathbf{Z}$ since the class of P depends only on its rank. Grothendieck's original definition had been for vector bundles over a space X and algebraists had generalized it by replacing vector bundles over X with projective modules over R. Thus K-theory seemed to divide into two camps, topological K-theory studying vector bundles and algebraic K-theory which studies rings. In both cases there is a set of functors mapping to abelian groups. The two theories are not very different and are often indistinguishable except for notation. Algebraic K-theorists often use topology in their work and topological K-theorists often use algebra. In fact, the most useful definition of higher algebraic K-groups is topological.

Although algebraic K-theory started as a set of functors from the category of rings to the category of abelian groups, it was soon generalized to a set of functors from an abelian category to abelian groups. Recently it has been generalized even further.

The Grothendieck group of R is $K_0(R)$. For each $i \geq 1$, the negative K-group, $K_{-i}(R)$, is a certain direct summand of K_0 of Laurent polynomials in i variables over R. Bass defined $K_1(R)$ to be $GL(R)/E(R)$, where $GL(R)$ is the group of all invertible matrices with entries in R and $E(R)$, the elementary group, is the subgroup of $GL(R)$ generated by $E_{ij}(r)$, $i \neq j$, $r \in R$, where $E_{ij}(r) = (a_{kl})$ with $a_{kk} = 1$, $a_{ij} = r$ and $a_{kl} = 0$ otherwise. Algebraically, K_1 studies how far the general linear group differs from the product of diagonal matrices and elementary matrices (they are equal for fields, division rings and local rings). In many ways K_1 is an attempt to generalize parts of linear algebra, especially the notions of dimension and determinant, to projective modules over an arbitrary ring. Milnor defined $K_2(R)$ which studies the relations in $E(R)$. Thus, K_1 and K_2 together determine the relations in the general linear group.

Some ring homomorphisms or sets of homomorphisms generate long exact sequences in K-theory, including Mayer-Vietoris sequences, localization sequences and the sequence of an ideal. These sequences were known for K_i, $i \leq 2$. Furthermore, in topological K-theory, there were definitions of $K_i^{\mathrm{top}}(X)$, $i \geq 3$, that extended these sequences. Therefore, in the late sixties many people tried to define higher algebraic K-groups that would extend these sequences and agree with higher topological K-groups when appropriate. In the early