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THE ERGODIC THEORETICAL PROOF OF SZEMERÉDI'S THEOREM

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Introduction. In 1975, E. Szemerédi proved the following theorem conjectured some forty years earlier by Erdös and Turan:

THEOREM I. *Let $\Lambda \subset Z$ be a subset of the integers of positive upper density, then Λ contains arbitrarily long arithmetic progressions.*

Partial results were obtained previously by K. F. Roth (1952) who established the existence of arithmetic progressions of length three in subsets of Z of positive upper density, and by E. Szemerédi (1969) who proved the existence of progressions of length four.

In 1976 Furstenberg noticed that the statement of Theorem I is equivalent to a statement about "multiple recurrence" of measure-preserving transformations, namely

THEOREM II. *Let $(X, \mathfrak{B}, \mu)$ be a probability measure space, let T be an invertible, measure-preserving transformation on $(X, \mathfrak{B}, \mu)$, and let $A \in \mathfrak{B}$ be a set of positive measure. Then for any positive integer k, there exists a subset $B \subset A$ with $\mu(B) > 0$ and an integer $n \geqslant 1$ with*

$$T^n B \subset A, T^{2n} B \subset A, \ldots, T^{(k-1)n} B \subset A$$

or what amounts to the same,

$$\mu\left(\bigcap_{j=0}^{k-1} T^{-jn} A\right) > 0.$$

It turned out to be possible to give an ergodic theoretic proof of Theorem II, thereby providing a new proof of Szemerédi's theorem.

Various elements of Furstenberg's original proof were simplified by Katznelson and Ornstein, and making use of this it became possible to prove a generalization of Theorem II with $T, T^2, \ldots, T^k$ replaced by any commuting set of measure-preserving transformations (cf. [**FK**]). This result leads to an analogue of Szemerédi's theorem for Z^r and, in fact, this proof proceeding by way of ergodic theory is the only one available so far for this analogue.

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