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SINGULARITIES OF SOLUTIONS OF SOME SCHRÖDINGER EQUATIONS ON $\mathbf{R}^n$

BY ALAN WEINSTEIN AND STEVEN ZELDITCH[1]

The Schrödinger equation for the wave function $\psi(t, x_1, x_2, \ldots, x_n)$ for a system of n one-dimensional oscillators is (in suitable units and coordinates)

$$i\hbar \frac{\partial \psi}{\partial t} = H_0 \psi \quad \text{where } H_0 = \sum_{j=1}^{n} \left(\frac{-\hbar^2}{2} \frac{\partial^2}{\partial x_j^2} + \frac{\omega_j}{2} x_j^2 \right).$$

The fundamental solution (or "propagator") $k_0(t, x, y)$ of this equation, with initial condition $k_0(0, x, y) = \Pi_{j=1}^n \delta(x_j - y_j)$, has singularities which can easily be determined by using a well-known explicit formula for k_0 [2, 6] for $t \neq m\pi/\omega_j$ and taking distribution limits as $t \longrightarrow m\pi/\omega_j$.

The result is:

(i) if $l_j = \omega_j t/\pi$ is not an integer for any j, then $k(t, x, y)$ is smooth in x;

(ii) if $l_j \in \mathbf{Z}$ for $j = j_1, \ldots, j_r$ then $k(t, x, y)$ is a smooth δ-function supported on the $(n - r)$-dimensional plane $\{x_{j_s} = (-1)^{l_{j_s}} y_{j_s}\}$.

In quantum mechanics, $k(t, x, y)$ is the (probability) amplitude that a particle *certainly* at y at time 0 has arrived at x at time t. Semi-classically, $k(t, x, y)$ describes a swarm of classical particles with initial positions all at y and with initial momenta uniformly distributed through $\mathbf{R}^n$. The occurrence of singularities in $k(t, x, y)$ coincides with the appearance of an "infinite density" of classical particles at a given point.

In this announcement, we show how this description of singularities can be extended to the case where the Hamiltonian is $H_0 + V$, V being (multiplication by) a potential function on $\mathbf{R}^n$ which is assumed to belong to the symbol class $S^0(\mathbf{R}^n)$; i.e. $|\partial_x^\alpha V| = O(\langle x \rangle^{-|\alpha|})$ for all multi-indices $\alpha \geqslant 0$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$. Semi-classically, the swarm is now additionally influenced by the force $-\text{grad } V(x)$; since this force approaches 0 at ∞, the higher energy particles in the swarm are little influenced by the perturbation, and so they will tend to re-accumulate at the same points as before. It turns out that these points also form the singular locus for the perturbed wave functions, demonstrating the stability of the picture described in (i), (ii) above.

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