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# PROBLEMS ON ABELIAN FUNCTIONS AT THE TIME OF POINCARÉ AND SOME AT PRESENT[1]

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This is an expanded version of our symposium lecture; it consists of two parts. In the first part we have tried to explain the problems on abelian functions at the time of Poincaré with a brief follow-up; in the second part we have explained, among others, a problem of Riemann and Weil on Jacobi's formula as one of the problems on abelian functions at present.

## 1. Abelian functions by Poincaré.

1-1. If the variable $x$ and a general solution $y$ of a linear differential equation with polynomial coefficients are algebraically dependent, the periods of abelian integrals of the first kind associated with the curve $f(x, y) = 0$ satisfy certain relations. In his earliest works on abelian functions Poincaré examined such relations in some special cases. He also used a similar relation in a joint paper with Picard of 1883 on a "theorem of Riemann". Poincaré later developed a general theory of reducible integrals. This theory played some role in almost all of his works on abelian functions. We shall start by recalling the theorem of Riemann:

There are three related theorems concerning a complex torus. If $f$ is a meromorphic function on $\mathbf{C}^g$, an element $a$ of $\mathbf{C}^g$ such that $f(z + a) = f(z)$ for a variable $z$ in $\mathbf{C}^g$ is called a period of $f$; the set of all periods of $f$ forms a closed subgroup of $\mathbf{C}^g$, called the period group of $f$. Let $\Lambda$ denote a lattice in $\mathbf{C}^g$, i.e., a discrete subgroup of $\mathbf{C}^g$ with compact quotient; then a meromorphic function $f$ on $\mathbf{C}^g$ whose period group contains $\Lambda$, i.e., a meromorphic function on the complex torus $T = \mathbf{C}^g/\Lambda$ considered as a function on $\mathbf{C}^g$, is called an *abelian function* relative to $\Lambda$ and a holomorphic function $\Theta$ on $\mathbf{C}^g$ with the property

$$\Theta(z + a) = \mathbf{e}(L_a(z))\Theta(z)$$

for every $a$ in $\Lambda$, in which $\mathbf{e}(t)$ stands for $\exp(2\pi\sqrt{-1}\,t)$ and $L_a(z)$ is an affine linear function of $z$ depending on $a$, is called a *theta function* also relative to $\Lambda$.

Finally a complex $g$-by-$2g$ matrix $\omega$ is called a *Riemann matrix* of degree $g$ if there exists a skew-symmetric integral matrix $C$ of degree $2g$, called a *principal matrix*, such that

$$\omega C^t\omega = 0, \qquad (1/2\sqrt{-1})\omega C^t\bar{\omega} > 0;$$

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