BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 6, Number 1, January 1982

THE UNDECIDABILITY OF THE RECURSIVELY ENUMERABLE DEGREES

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Let $\leqslant$ be Turing reducibility between subsets of ω and let $\mathcal{R}$ be the collection of all recursively enumerable subsets of ω. For A in $\mathcal{R}$, $\mathbf{A}$ is the Turing degree of A; and $\mathbf{R} = \{\mathbf{A}; A \text{ is in } \mathcal{R}\}$. (See [2] for any unexplained notions or notation used above, or below). We also identify $\mathbf{R}$ with the structure $\langle \mathbf{R}, \leqslant \rangle$.

We wish to announce the result

THEOREM. *The first order theory of* $\mathbf{R}$ *is undecidable.*

To prove this we show that the theory of partial orderings is reducible to the theory of $\mathbf{R}$, as follows.

Recall, $\mathbf{R}$ is an upper-semi-lattice; thus for a, b in $\mathbf{R}$, their join $a \vee b$ is in $\mathbf{R}$.

LEMMA. *For* $\mathcal{P}$ *a partial ordering recursive in* (*say*) $0'$, *there are* a, b_p $(p \in \mathcal{P})$, c, d, e *in* $\mathbf{R}$ *such that*

(i) $b_p \leqslant a$,

(ii) $a \nleqslant (b_p \vee C)$,

(iii) *for each* $Z \leqslant a$, *either*

(α) $a \leqslant (Z \vee c)$, *or* $\exists\, p \in \mathcal{P}$ *such that*

$(\beta)_p$ $Z \leqslant (b_p \vee d)$,

(iv) *for* $p \neq q$, $b_p \nleqslant (b_q \vee d)$,

($\sim$) *for* p, q *in* $\mathcal{P}$, $p \leqslant q$ *iff* $b_p \leqslant (b_q \vee d \vee e)$.

Now, for a, b, c, d in $\mathbf{R}$ let $\phi(a, b, c, d) \equiv (b \leqslant a)$ and $(a \nleqslant (b \vee c))$ and $(\neg \exists Z\ (Z \leqslant a$ and $a \nleqslant (Z \vee c)$ and $(Z \vee d) > (b \vee d)))$. For a, c, d, e in $\mathbf{R}$, let $Q(a, c, d, e) = \{b \vee d \vee e; \mathbf{R} \models \phi(a, b, c, d)\}$. For a, c, d, e as in the lemma, $Q(a, c, d, e) = \{b_p \vee d \vee e; p \in \mathcal{P}\}$, and $\langle Q(a, c, d, e), \leqslant \rangle$ is isomorphic to $\mathcal{P}$.

Thus for ψ a sentence of the language of partial orderings: ψ is true of some partial ordering iff (by the usual proof of the completeness theorem) $\mathcal{P} \models \psi$ for some $\mathcal{P}$ recursive in $0'$ iff (by the lemma) $\exists a, c, d, e$ in $\mathbf{R}$ $(\langle Q(a, c, d, e), \leqslant \rangle \models \psi)$.

The lemma is proven, of course, by a priority argument. The type of priority argument used can best be described as an infinite injury argument with a finite injury priority argument on top of it. This kind of construction was first

Received by the editors February 3, 1981.

AMS (*MOS*) *subject classifications* (1970). Primary 02F25, 02F30.

[1] The authors were supported by the NSF.