BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 6, Number 1, January 1982

WEB GEOMETRY

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Introduction. Poincaré published two papers on surfaces of translation [**10, 11**].[2] They were among his lesser known papers. In the following pages I wish to show that the subject he touched is an exciting one and deserves further investigation.

1. Lie's theorem on surfaces of double translation and its developments. A surface M of translation in R^3 is defined by the parametric equations

$$x^\lambda = f^\lambda(u) + g^\lambda(v), \qquad 1 \leqslant \lambda \leqslant 3, \tag{1}$$

where x^λ are the coordinates in R^3 and f^λ, g^λ are arbitrary smooth functions. It is immediately seen that the tangent lines to the u-curves (respectively the v curves) are independent of v (resp. u) and define a curve C_u (resp. C_v) in the plane at infinity.

M is called a *surface of double translation* if it is a surface of translation in a second way, i.e., given also by the equations

$$x^\lambda = h^\lambda(s) + k^\lambda(t), \qquad 1 \leqslant \lambda \leqslant 3, \tag{2}$$

such that exactly two of the equations

$$f^\lambda(u) + g^\lambda(v) - h^\lambda(s) - k^\lambda(t) = 0, \qquad 1 \leqslant \lambda \leqslant 3, \tag{3}$$

are independent. In 1882 Sophus Lie proved the remarkable theorem [**7**]:

If *M is a surface of double translation* in R^3, *the four curves C_u, C_v, C_s, C_t in the plane at infinity defined by the tangent lines to the four families of parametric curves belong to the same algebraic curve of degree four.*

The theorem means that the solutions of the functional equations (3) on a surface arise from an algebraic structure. Lie's proof makes use of the integrability conditions of over-determined systems of partial differential equations. In fact, from (1) we have

$$\partial^2 x^\lambda / \partial u \partial v = 0, \tag{4}$$

which means that the parametric curves form a conjugate net, i.e., their tangent directions separate harmonically the asymptotic directions at each point. If the surface M is given in the nonparametric form

$$z = z(x, y), \tag{5}$$

Received by the editors November 12, 1980.

1980 *Mathematics Subject Classification.* Primary 53A60, 14D25.

[1]Work done under partial support of NSF grant MC577-23579.

[2]Numbers in brackets refer to the Bibliography at the end of the paper.

0002-9904/82/0000-0243/$03.00