BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 5, Number 3, November 1981

0002-9904/81/0000-0505/$02.25

BOOK REVIEWS

Inequalities: Theory of majorization and its applications, by Albert W. Marshall and Ingram Olkin, Mathematics in Science and Engineering, Vol. 143, Academic Press, New York, 1979, xx + 569 pp., $49.50.

Probability inequalities in multivariate distributions, by Y. L. Tong, Probability and Mathematical Statistics, A Series of Monographs and Textbooks, Academic Press, New York, 1980, xiii + 239 pp., $ 29.50.

Both monographs make extensive use of a (quasi) partial ordering of R^n called majorization and the corresponding class of (Schur) increasing functions on R^n. In this connection, it is best to think of $x = (x_1, \ldots, x_n) \in R^n$ as representing the measure μ on the reals R of total mass n which is defined by $\mu(A) = \sum_{x_i \in A} 1$. Let ν denote the analogous measure represented by the point $y \in R^n$. One says that x is majorized by y and also that $x \prec y$ or that y is a dilation of x when

$$(1) \qquad \sum_{i=1}^{n} f(x_i) = \int f\, d\mu \leqslant \int f\, d\nu = \sum_{i=1}^{n} f(y_i)$$

holds for each convex function f on R. It implies that μ and ν have the same mass and the same centre of gravity. A necessary and sufficient condition is that

$$(2) \qquad \sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]} \quad \text{for } k = 1, 2, \ldots, n,$$

insisting that the equality sign holds when $k = n$. Here, $x_{[1]} \geqslant \cdots \geqslant x_{[n]}$ are the x_i arranged in decreasing order and, similarly, $y_{[1]} \geqslant \cdots \geqslant y_{[n]}$. If (1) is only required for the increasing (decreasing) convex functions on R then one speaks of weak sub-majorization $x \prec_w y$ (or weak super-majorization $x \prec^w y$, respectively). The first is equivalent to (2).

Let $\mathcal{A}$ be an open convex subset of R^n which is symmetric, that is, invariant under each permutation of the coordinates. A function $\phi\colon \mathcal{A} \to R$ is said to be Schur increasing (or Schur convex) if it is nondecreasing relative to the partial ordering $x \prec y$ of $\mathcal{A}$; similarly for Schur decreasing functions, also called Schur concave functions. A Schur increasing function is always symmetric. An obvious example would be

$$(3) \qquad \phi(x) = \phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} f(x_i),$$

with $f\colon R \to R$ a convex function. More generally, every symmetric and convex (concave) function on $\mathcal{A}$ is Schur convex (Schur concave). A symmetric C^1 function ϕ on $\mathcal{A}$ is Schur increasing if and only if $\phi_{(i)}(x) - \phi_{(j)}(x)$ is