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ON THE HOPF INDEX THEOREM AND THE HOPF INVARIANT[1]

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Let $f: N \longrightarrow M$ be a C^∞ map of oriented compact manifolds, and let L be an oriented closed submanifold of codimension $q \geqslant 1$ in M. If w is a closed form Poincaré dual to L, we show that $f^{-1}L$, with multiplicities counted, is Poincaré dual to f^*w in N and is even meaningful on a "secondary" level. This leads to generalized versions of the Hopf invariant, the Hopf index theorem and the Bezout theorem.

We assume that the connected components $\Gamma_1, \ldots, \Gamma_l$ of $f^{-1}L$ are submanifolds of codimension q in N. Let ord Γ_i be the intersection number of L and $f|B$, where B is a q-dimensional submanifold meeting Γ_i transversally at a single point. A proper choice of orientations makes ord $\Gamma_i \geqslant 0$.

THEOREM 1. *The cycle $\Sigma(\text{ord } \Gamma_i)\Gamma_i$ is Poincaré dual to f^*w.*[2]

This assertion improves a known theorem, which requires that f is transversal to L and, consequently, ord $\Gamma_i = 1$.

THEOREM 2. *Let w' be an integral closed q'-form on M with $q + q' - 1 > \dim M$. If both f^*w and f^*w' are exact with $f^*w' = du$ on N and if σ is a closed p-form on N with $p + q + q' = \dim N$, then*

$$\int_N f^*w \wedge u \wedge \sigma = \sum (\text{ord } \Gamma_i) \int_{\Gamma_i} u \wedge \sigma.$$

COROLLARY. *Let $f: N \longrightarrow M$ be an arbitrary C^∞ map (without any condition on $f^{-1}L$). If r is the least positive integer making $rH_{q+q'-1}(N; Z)$ free abelian, then the cohomology class of $rf^*w \wedge u$ is integral.*

A sketched proof of Theorem 1 runs as follows. There exists a $(q-1)$-form v on $M - L$ with $dv = w|M - L$. Let σ be a closed p-form on N such that

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[2] The author wishes to thank the referee for pointing out the general validity of Theorems 1 and 2. Our proof of these theorems is a modification of the original version, which, however, already suffices to cover Theorem 3 and other applications in this note.

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