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LINEAR GROUPS OF FINITE COHOMOLOGICAL DIMENSION

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Our main result provides necessary and sufficient conditions for a finitely-generated subgroup of $GL_n(\mathbf{C})$, $n > 0$, to have finite virtual cohomological dimension. A group has finite virtual cohomological dimension (VCD) if it has a subgroup of finite index which has finite cohomological dimension; this dimension is, in fact, the same for all torsion-free subgroups of finite index. It is, of course, necessary for a group Γ with $\mathrm{VCD}(\Gamma) < \infty$ to have torsion-free subgroups of finite index; this is guaranteed in the case of finitely-generated linear groups by a well-known result of Selberg which extends ideas of Minkowski.

A subgroup of $GL_n(\mathbf{C})$ is called unipotent if it is contained in a conjugate of the group of upper triangular matrices with all diagonal entries equal to one. Any unipotent subgroup is nilpotent; hence, a finitely-generated unipotent subgroup is polycyclic and torsion-free. It is well known that a polycyclic group has finite cohomological dimension if and only if it is torsion-free; moreover, the cohomological dimension is the same as the Hirsch rank. For a solvable group Γ with solvable series, $1 = \Gamma_n < \Gamma_{n-1} < \cdots < \Gamma_1 = \Gamma$, the Hirsch rank, $h(\Gamma) = \Sigma_{i=1}^{n-1} \dim_{\mathbf{Q}}(\Gamma_i/\Gamma_{i+1} \otimes \mathbf{Q})$, is independent of the choice of solvable series; thus, for a polycyclic group Γ, $h(\Gamma)$ is the number of infinite factors in a normal series with cyclic quotients.

We announce our main result.

THEOREM. *Let A be a finitely-generated integral domain of characteristic zero. A group $\Gamma \subset GL_n(A)$, $n > 0$, has finite VCD if and only if there is a finite upper bound on the Hirsch ranks of its finitely-generated unipotent subgroups.*

We obtain easily the following curious corollary.

COROLLARY 1. *Every finitely-generated subgroup of the unitary group $U_n(\mathbf{C})$, $n > 0$, has finite virtual cohomological dimension.*

The following result is immediate; it, however, was original motivation for our Theorem.

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