BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 4, Number 2, March 1981

VON NEUMANN REGULAR RINGS: CONNECTIONS WITH FUNCTIONAL ANALYSIS

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Most ring theorists and functional analysts are at least vaguely aware that von Neumann invented regular rings 45 years ago in connection with certain operator algebras, but that connection has grown rusty and pretty much disused as regular rings and operator algebras have gone their separate ways. My purpose here is to report on several more recent connections between regular rings and functional analysis, developed in the past decade, through which each subject has made a positive contribution to the other.

The first three sections of this report are ancient history, sketching the original development of regular rings in connection with continuous geometries, von Neumann algebras, and AW^*-algebras, and providing some of the relevant ring and operator algebra concepts for those readers who don't have them right at hand. The remaining three sections sketch the recent connections, including some of the ways in which regular rings have aided the study of Rickart C^*-algebras and approximately finite-dimensional C^*-algebras, and some of the ways in which Choquet simplices have aided the study of regular rings.

I. Complemented modular lattices and regular rings. Regular rings were invented by von Neumann in the mid-1930's in order to provide an algebraic framework for studying the lattices of projections in the operator algebras he was investigating. This framework actually dealt with a remarkably large class of general lattices, including most complemented modular lattices, which were then thought to be the appropriate setting for the logical formalism of quantum mechanics.

Von Neumann modelled this framework on the coordinatization of classical projective geometry, which was at that time just being recast in a lattice-theoretic mold, by Birkhoff **[5]** and Menger **[18]**, among others. These authors viewed projective geometries as lattices L satisfying

(a) COMPLEMENTATION. The lattice L has a least element 0 and a greatest element 1, and every element $x \in L$ has at least one *complement*, namely an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$.

(b) MODULARITY. Whenever $x, y, z \in L$ with $x \leqslant z$, then $(x \vee y) \wedge z = x \vee (y \wedge z)$. (This is a weak form of the distributive law.)

This paper is an expanded version of an hour address given on January 3, 1980 at the Annual Meeting of the American Mathematical Society in San Antonio, Texas; received by the editors April 20, 1980.

1980 *Mathematics Subject Classification.* Primary 16A30, 06C20, 46L10, 46L05, 46A55.

0002-9904/81/0000-0100/$03.50