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FOR $n > 3$ THERE IS ONLY ONE FINITELY ADDITIVE ROTATIONALLY INVARIANT MEASURE ON THE n-SPHERE DEFINED ON ALL LEBESGUE MEASURABLE SUBSETS

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The following paragraph is taken from the introduction of Joseph Rosenblatt's paper **[R]**.

"Let β be the ring of Lebesgue measurable sets in the n-sphere S^n, and let λ_n denote the Lebesgue measure on β normalized by $\lambda_n(S^n) = 1$. The classical characterization by Lebesgue of λ_n is that it is the unique positive real-valued function μ on β which satisfies these three conditions:

(a) $\mu(S^n) = 1$;

(b) μ is invariant under isometries;

(c) μ is countably additive.

In 1923 Banach **[B]** studied the question of Ruziewicz whether μ is still unique when (c) is replaced by

(c_0) μ is finitely additive.

Banach gave a negative answer to this question for S^1 but for S^n, $n \geqslant 2$, the question is still unanswered."

From the body of Rosenblatt's paper one can extract the implication that *if Lebesgue measure* λ_n *on* S^n *is not characterized by* (a), (b), *and* (c_0) *then there is a net of measurable subsets* $(A_\alpha) \subset S^2$ *which is asymptotically invariant and nontrivial, namely* $\lim_\alpha(\lambda_n(gA_\alpha \Delta A_\alpha)/\lambda_n A_\alpha) = 0$ *for all rotations g and so that* $0 < \lambda_n(A_\alpha) \leqslant c < 1$ (Theorem 1.4 of **[R]**). Here $A \Delta B = A \cup B - A \cap B$.

The following Proposition will show that such asymptotically invariant nets on S^n are impossible, $n > 3$.

PROPOSITION. *For each $n > 3$ there is a countable subgroup Γ_n in the group O_{n+1} of rotations of S^n satisfying*

(i) *the action of Γ_n on S^n is ergodic,*

(ii) *the group Γ_n satisfies Kazhdan's property T:*

There exist a finite subset $\Lambda \subset \Gamma_n$ and an $\epsilon > 0$, so that for any unitary representation π if Γ, if there exists a vector ζ in H_π such that $\|\zeta\| = 1$,

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