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FOUR-DIMENSIONAL TOPOLOGY: AN INTRODUCTION

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Introduction. This paper is an elaboration of the talk I gave at the American Mathematical Society meeting held at Memphis State University in November, 1977. In that talk I tried to illustrate what was special about four-dimensional topology and mention some of the advances in that subject. I have expanded upon that talk somewhat in this paper in order to give a clearer picture of the geometric ideas and methods currently used in studying four-dimensional manifolds.

The basic problem of geometry is the characterization of spaces by means of algebraic invariants. In its simplest form it is the "Side-Angle-Side" theorem of high school geometry which characterizes triangles up to congruence or the 'Angle-Angle' theorem which characterizes them up to similarity. In the more rarefied realm of algebraic and geometric topology we are especially interested in attaching algebraic objects such as groups, rings, modules, etc., to a space M already known to be a compact connected manifold in such a way that if two manifolds M, M' have the same objects associated to them then they are isomorphic (where isomorphic might mean 'homeomorphic' or 'piecewise-linearly homeomorphic' or 'diffeomorphic'). As a result of the undecidability of the word problem for finitely-presented groups [**Bo**], [**BHP**] there is, in general, no finite procedure for deciding in all cases whether two groups given by finite sets of generators and relations are isomorphic. Furthermore, as is well known, any finitely-presented group can be realized as the fundamental group of a compact n-manifold if $n \geqslant 4$. Thus provided $n \geqslant 4$ we have no generally effective procedure for determining if M_1^n is isomorphic to M_2^n. (Henceforth we write '$=$' for isomorphic.)

Thus we must ask a more restricted question. In particular we can ask to what extent is M^n determined by its homotopy type? We thus consider the following question.

Let $\mathfrak{M}^n(X)$ denote the equivalence classes of pairs (M, f), where M is a compact smooth (or PL) n-manifold, $f\colon M \to X$ is a homotopy equivalence and $(M, f) \sim (M', f')$ if and only if there is an isomorphism $M \to M'$ making

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