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SINGULARITIES AND GROUP ACTIONS

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To my parents

1. Introduction. The emergence of Transformation Groups as a separate branch of mathematics about a hundred years ago is connected with the names of Sophus Lie and Felix Klein. The invariant theorists of the day asked to find all invariant polynomials of a given linear group G. Let us fix the field $\mathbf{C}$ and let f: $\mathbf{C}^{n+1} \to \mathbf{C}$ be a polynomial mapping. A related question asks for $\operatorname{Aut}(f)$, the group of linear transformations leaving f fixed. Call $\mathbf{z}$ a *critical point* of f if the partial derivatives $\partial f/\partial z_i$ vanish at $\mathbf{z}$. A critical point is called *isolated* if it is the only critical point in some neighborhood of $\mathbf{z}$. In the context of invariant theory f is homogeneous of degree m. If $m > 2$, then $\mathbf{0}$ is a critical point, and if it is isolated, then it is the only critical point of f. Lie [49] noted (without proof; for history and a proof see Orlik and Solomon [59]):

THEOREM. *If G is a linear group leaving a homogeneous polynomial f of degree $m \geqslant 3$ invariant, and the critical point of f at $\mathbf{0}$ is isolated, then G is finite. In particular,* $\operatorname{Aut}(f)$ *is finite.*

In his famous lectures on the icosahedron Klein [45] observed the connection between the binary icosahedral group G and the polynomial $f(z_0, z_1, z_2) = z_0^5 + z_1^3 + z_2^2$. The ring of invariant polynomials of a representation of G in $SU(2)$ is generated by three homogeneous polynomials of degrees 6, 10, 15. There is one polynomial dependence among them which (up to coefficients) reads $z_0^5 + z_1^3 + z_2^2 = 0$. Thus $\mathbf{C}^2/G$ is isomorphic to the hypersurface $V = f^{-1}(0)$ in $\mathbf{C}^3$. In fact V is the cone over Poincaré's dodecahedral space. The critical point $\mathbf{0}$ of f is isolated and it is also called an isolated singularity of V. Such a singular point has a (nonunique) *resolution*, consisting of a nonsingular algebraic variety $\tilde{V}$ and a proper map π: $\tilde{V} \to V$ such that the restriction π: $\tilde{V} - \pi^{-1}(\mathbf{0}) \to V - \mathbf{0}$ is an isomorphism. Finding a resolution is in general rather cumbersome. However, this polynomial has a symmetry which may be exploited, in that for $t \in \mathbf{C}$ we have $f(t^6 z_0, t^{10} z_1, t^{15} z_2) = t^{30} f(z_0, z_1, z_2)$, so V is invariant under the action of the multiplicative group of nonzero complex numbers, $\mathbf{C}^*$. We shall return to this in §7.

This survey will consider some highlights of the interaction between Transformation Groups and Singularities from the last decade. I wish to thank I. Dolgachev, A. Durfee, H. Hamm, L. Kauffman, W. Neumann, T. Petrie, R.

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