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Lectures on closed geodesics, by Wilhelm Klingenberg, Die Grundlehren der Math. Wissenschaften, Band 230, Springer-Verlag, Berlin, Heidelberg, New York, 1978, ix + 227 pp.

Poincaré's 1905 paper *Sur les lignes géodésiques des surfaces convexes* **[6]** considered several problems: (A) the existence of any closed geodesics on convex surfaces, (B) the existence of three distinct simple closed geodesics on surfaces of genus zero, (C) the manner in which families of closed geodesics vary as a surface changes analytically, and (D) the stability of closed geodesics. In the succeeding three-quarters of a century, solutions or outlines of solutions of these problems have provided work for many mathematicians, including such famous men as G. D. Birkhoff, L. Lyusternik, and Marston Morse. And the theory seems to have more than its share of pitfalls: the pioneers published some proofs which did not meet the standards for rigor of later generations, and various ideas have distinguished genealogies of error. (Smale, [**8**, p. 691], cites one case.)

An appreciation of some of the elements involved may be gained by looking at problem (A). Let PM be the space of piecewise differentiable maps, c, of the circle into M, a surface diffeomorphic to S^2. The energy function

$$E(c) = \frac{1}{2}\int_0^1 |\dot{c}(t)|^2\, dt$$

has closed geodesics and constant maps as its critical points. Since PM is connected, one cannot find nontrivial closed geodesics merely by minimizing E. But PM is not simply-connected, so there is hope for finding a critical point by first maximizing E on certain homotopically nontrivial curves in PM, then minimizing the result over a fixed homotopy class. This is the essence of the minimax method, and even in this lowest dimension requires considerable work to make rigorous. However, this approach is the heart of the theory.

One recognizes a preliminary division of labor: the study of the functional E, and the study of the topology of PM. These problems were originally attacked by using finite-dimensional approximations to PM, so that E could be treated by ordinary calculus (or "finite-dimensional Morse theory"). Now, with the language of infinite-dimensional manifolds and fibrations available, the exposition appears much cleaner, but the difficulty remains. PM is often replaced by the Hilbert manifold, $\bigwedge M$, of absolutely continuous maps with square-summable first derivatives. Stepwise deformations are replaced by a flow; for example, along the gradient of E, when that is defined.

With suitable modifications, the process outlined above works for any dimension and the result is the theorem of Lyusternik and Fet (1951): On every compact Riemannian manifold, there exists at least one closed geodesic. Thus the natural generalization of problem (A) has received a satisfactory answer.

Problem (B) was solved by Lyusternik and Schnirelmann in 1929, using the category theory they invented for the purpose. In fact, they showed that there