BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 5, September 1976

BOUNDS ON THE EIGENVALUES OF THE LAPLACE AND SCHROEDINGER OPERATORS

BY ELLIOTT LIEB[1]

Communicated by I. M. Singer, April 19, 1976

If Ω is an open set in $\mathbf{R}^n$, and if $\widetilde{N}(\Omega, \lambda)$ is the number of eigenvalues of $-\Delta$ (with Dirichlet boundary conditions on $\partial\Omega$) which are $\leqslant \lambda$ ($\lambda \geqslant 0$), one has the *asymptotic* formula of Weyl [1], [2]: $\lim_{\lambda\to\infty} \lambda^{-n/2}\widetilde{N}(\Omega, \lambda) = C_n|\Omega|$. Here $|\Omega|$ is the volume of Ω and $C_n = (4\pi)^{-n/2}\Gamma(1 + n/2)^{-1}$. The same holds [3] if $\mathbf{R}^n$ is replaced by a Riemannian manifold, M, with $|\Omega|$ being the Riemannian volume and Δ being the Laplace-Beltrami operator. One purpose of this note is to state that there often exist bounds of the form

$$\widetilde{N}(\Omega, \lambda) \leqslant D_n\lambda^{n/2}|\Omega|, \forall\lambda \geqslant 0, \forall\Omega \subset M, \tag{1a}$$

$$\widetilde{N}(\Omega, \lambda) \leqslant (D_n\lambda^{n/2} + E_n)|\Omega|, \quad \forall\lambda \geqslant 0, \forall\Omega \subset M, \tag{1b}$$

with D_n, E_n *independent of* λ *and* Ω and depending only on M. (1a) holds for noncompact M if condition (8), below, holds. In particular, (1a) holds for $\mathbf{R}^n$ and for homogeneous spaces with curvature $\leqslant 0$. (1b) always holds for compact M, and it also holds for noncompact M if condition (9) holds.

REMARK. There is an asymptotic formula [4], [5]: $\widetilde{N}(\Omega, \lambda) = C_n\lambda^{n/2}|\Omega| + O(\lambda^{(n-1)/2})$. While this has the correct limiting constant, C_n, the remainder, $O(\cdot)$, can get very large if Ω is very irregular. The remainder is not bounded by a universal constant times $|\Omega|\lambda^{(n-1)/2}$ or even $|\Omega|\lambda^{n/2}$. Our emphasis is different. By introducing $D_n \geqslant C_n$ we have a bound which is universal in the sense that it depends on M but *not* on $\Omega \subset M$; in particular, (1) is applicable to unbounded Ω.

A second, closely related problem is to estimate $N_\alpha(V)$ = number of nonpositive eigenvalues of the Schroedinger operator $-\Delta + V(x)$ on $L^2(M)$ which are $\leqslant \alpha \leqslant 0$. There exists an asymptotic formula [6], [7], [8] for suitably regular V:

$$\lim_{\gamma\to\infty} \gamma^{-n/2}N_{\gamma\alpha}(\gamma V) = C_n \int_M [V(x) - \alpha]_-^{n/2}\, dx \tag{2}$$

AMS (MOS) subject classifications (1970). Primary 58G99, 35J05, 35J10, 35P15, 35P20; Secondary 47F05, 81A09, 81A45.

[1] Work supported by U. S. National Science Foundation grant MCS 75-21684.

Copyright © 1976, American Mathematical Society