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# RESEARCH ANNOUNCEMENTS

## THE PRINCIPAL SYMBOL OF A DISTRIBUTION

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In Hörmander's theory of Fourier integral operators [1], a principal symbol is constructed for a certain class of distributions in such a way that, when the construction is applied to the Schwartz kernel of a pseudodifferential operator, one obtains the usual principal symbol of the operator. In this note, we describe a generalization of Hörmander's construction which may be applied to an arbitrary distribution on a manifold. Details will appear in [4].

1. **Local definition and invariance properties.** For a complex vector space $V$, we define $V$-valued distributions on $\mathbf{R}^n$ by taking as test functions objects of the form $u = u(x)\,dx$, where $u(x)$ is a compactly supported $C^\infty$ function with values in $V^*$, and $dx$ is the density $|dx_1 \wedge \cdots \wedge dx_n|$. For $\tau > 0$, we define $u_\tau$ to be $u(\tau x)\,dx$. If $g$ is a $V$-valued distribution, and $\varphi$ is a $C^\infty$ function with $\varphi(0) = 0$, we define the family $\{g_\tau^\varphi\}_{\tau > 0}$ of distributions by

$$\langle g_\tau^\varphi, u\rangle = \langle g, e^{-i\tau\varphi} u_{\sqrt{\tau}}\rangle. \tag{1}$$

For $N \in \mathbf{R}$, we write $g_\tau^\varphi \in O(\tau^N)$ if $\tau^{-N} g_\tau^\varphi$ remains bounded in distribution space [3] as $\tau \longrightarrow \infty$.

LEMMA. *For every $g$ and $\varphi$, $g_\tau^\varphi \in O(\tau^N)$ for some $N \in \mathbf{R}$.*

DEFINITION. $\inf\{N | g_\tau^\varphi \in O(\tau^N)\} \in [-\infty, \infty)$ is called *the order of $g$ at $\varphi$* and denoted by $O_\varphi(g)$.[2]

THEOREM 1. (a) *If $O_\varphi(g) \leqslant N$ and $\psi(x) = \varphi(x) + \Sigma a_{jk} x_j x_k + O(x^3)$, then $g_\tau^\psi - e^{-i\Sigma a_{jk}x_j x_k} g_\tau^\varphi \in O(\tau^{N-1/2})$.*

(b) *If $O_\varphi(g) \leqslant N$ and $A$ is a $C^\infty$ function with values in* $\mathrm{Hom}(V, V)$, *then $(Ag)_t^\varphi - A(0) g_\tau^\varphi \in O(\tau^{N-1/2})$.*

(c) *If $O_\varphi(g) \leqslant N$ and $\theta\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is a diffeomorphism with $\theta(0) = 0$, then $(\theta^* g)_\tau^{\theta^*\varphi} - (T_0\theta)^*(g_\tau^\varphi) \in O(\tau^{N-1/2})$.*

DEFINITION. If $O_\varphi(g) \leqslant N$, the class of $\tau^{-N} g_\tau^\varphi$ modulo $O(\tau^{-1/2})$ is called the *principal symbol* of order $N$ for $g$ at $\varphi$.

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[2] See final note added in proof.