BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 1, January 1976

CONFORMAL MAPS ON HILBERT SPACE

BY MELVYN HUFF

Communicated February 15, 1975

1. **Introduction.** In [1] Nevanlinna gave a simple proof of the following theorem of Liouville. (Precise definitions appear below.)

THEOREM 1. *Suppose U is a connected open set in a real Hilbert space H of dimension $\geqslant 3$ (including ∞) and $f\colon U \longrightarrow H$ is C^4 and conformal. Then f is either*

(a) *an affine map whose linear part is a constant multiple of a unitary operator,*

(b) *an inversion with respect to a sphere,*

(c) $f_1 \circ f_2$ *where f_1 is of type* (a) *and f_2 is of type* (b).

REMARKS. (i) The dimension of H must be $\geqslant 3$ because every holomorphic map on $\mathbf{C}$ with a nowhere zero derivative is conformal.

(ii) For $\mathbf{R}^n$, the theorem is known even for f just C^1 [2].

(iii) The proof of Nevanlinna depends on f being C^4.

In this paper we outline how a technique in [3], when recognized as applying to conformal mappings and suitably modified, can be used to prove the theorem with f only C^3.

2. **Notation and definitions.** H will be a real infinite dimensional Hilbert space and U a connected open subset. A map is C^n if it is n times continuously Fréchet differentiable as in [4]. A C^1 function $f\colon U \longrightarrow H$ is called conformal if Df_x is a linear isomorphism and there is a function $c\colon U \longrightarrow \mathbf{R}$ such that

$$\langle Df_x(h_1), Df_x(h_2)\rangle = c(x)\,\langle h_1, h_2\rangle$$

for all x in U and all h_1, h_2 in H. (This definition is merely a reformulation of the more geometric definition that says f preserves the angle between two curves meeting at a point.) Banach and Hilbert manifolds are defined as in [4].

By an inversion with respect to the sphere $\{x \in H\colon \|x - p\| = r\}$ I mean the map $x \longrightarrow x'$ where

(i) $\|x - p\|\,\|x' - p\| = r^2$ and

(ii) x and x' lie on the same ray originating at p. The analytic form of such an inversion is

$$x \longrightarrow r^2(x - p)\|x - p\|^{-2} + p.$$

AMS (MOS) subject classifications (1970). Primary 58B10, 58B20, 46C10.