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THE FAILURE OF SPECTRAL ANALYSIS IN L^p FOR $0 < p < 1$

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1. **Introduction.** For $0 < p < 1$, L^p is the space of measurable f on the circle group $\mathbf{T}$ with

$$\|f\|_p = \left[(2\pi)^{-1} \int_{-\pi}^{+\pi} |f(x)|^p \, dx\right]^{1/p} < \infty.$$

If $0 < p < 1$, L^p is not a Banach space, but is a metric space with distance defined by $d(f, g) = \|f - g\|_p^p$.

A linear subspace of L^p will be called a $\mathbf{T}$-*subspace* if and only if it is closed and translation invariant. If F is a function or a collection of functions in L^p, then $L^p(F)$ will denote the smallest $\mathbf{T}$-subspace of L^p containing F, the $\mathbf{T}$-subspace of L^p *generated* by F. If $F = \{e^{in\cdot}: n \in \Delta\}$, is a collection of exponential functions, $L^p(F)$ will also be denoted by $L^p(\Delta)$.

For $p \geqslant 1$, the classification of the $\mathbf{T}$-subspaces of L^p is straightforward (see [3, Chapter 11]). The map

$$\Delta \rightsquigarrow L^p(\Delta) \tag{1.1}$$

gives a 1-1 correspondence between the collection of all subsets of integers and all $\mathbf{T}$-subspaces of L^p.

The purpose of this note is to point out that the case $0 < p < 1$ is much more intricate, to be specific, the map (1.1) is neither 1-1 nor onto. We shall outline proofs of results which imply the following.

THEOREM 1. *Let* $0 < p < 1$. *Then*

(i) L^p *has nontrivial* $\mathbf{T}$-*subspaces containing no exponentials*;

(ii) *There are distinct sets* Δ *and* Γ *of integers with* $L^p(\Delta) = L^p(\Gamma)$.

Details will be published elsewhere. In what follows, "Proof" should of course be interpreted to mean "Outline of Proof".

2. **Spectral analysis in H^p for $0 < p < 1$; Cauchy integrals.** Here we restrict to the $\mathbf{T}$-subspace $L^p(\{e^{in\cdot}: n \geqslant 0\})$, which is denoted by H^p. (For the basic properties of H^p which we use in what follows, see [2, Chapter 7], [4, Chapter 3] or [1].) H^p can also be characterized as follows: Let $\mathbf{D}$ be the unit disk $\{z: |z| < 1\}$. We define $H^p(\mathbf{D})$ to consist of all functions F which are analytic in D with $|||F|||_p = \sup\{\|F_r\|_p: 0 < r < 1\} < \infty$, where each F_r is de-

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