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ON THE R-FORMS OF CERTAIN ALGEBRAIC VARIETIES

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In this note we determine explicitly the inequivalent models over $\mathbf{R}$ of certain varieties $U = \Gamma\backslash H^n$ where Γ is a unit group of a totally indefinite quaternion algebra over a totally real number field k, $|k : \mathbf{Q}| = n$, and $H =$ upper half-plane. For each model defined over $\mathbf{R}$ we give a formula for the number of connected components of the manifold of real points of U.

1. Let A be a totally indefinite division quaternion algebra over a totally real number field k, $\mathfrak{G}$ a maximal order in A, and $\Gamma = \{\gamma \in \mathfrak{G}^\times$ with reduced norm $\nu(\gamma) = 1\}$. We fix an isomorphism λ: $A_{\mathbf{R}} = A \otimes_{\mathbf{Q}} \mathbf{R} \simeq M_2(\mathbf{R})^n$, $n = |k : \mathbf{Q}|$; then $\lambda(\Gamma \otimes 1) \subset \mathrm{SL}_2(\mathbf{R})^n$ and thus $\Gamma/\pm 1$ acts properly discontinuously on $H^n =$ product of n copies of the upper half-plane via fractional linear transformations. Under certain assumptions on A, $\Gamma/\pm 1$ will act without fixed points so that $U = \Gamma\backslash H^n$ will be a compact complex manifold. It is well known that such U are imbeddable as nonsingular complex projective algebraic varieties.

A real model of U is a pair (U', φ) consisting of a nonsingular projective variety $U' \subset \mathbf{P}^N(\mathbf{C})$ defined over $\mathbf{R}$ and a biholomorphic map φ: $U \longrightarrow U'$. Two real models are equivalent if there exists a biregular isomorphism f: $U_1' \longrightarrow U_2'$ with f defined over $\mathbf{R}$. An equivalence class of real models will be called an $\mathbf{R}$-form of U. To each real model (U', φ) of U we associate an antiholomorphic involution ρ: $U \longrightarrow U$ by the formula $\rho(x) = \varphi^{-1}(\overline{\varphi(x)})$. We call the points $x \in U$ such that $x = \rho(x)$ the real points of the model (U', φ). The following is well known:

LEMMA 1. *The* $\mathbf{R}$*-forms of* U *are in one-to-one correspondence with the* $\mathrm{Aut}^h(U)$ *conjugacy classes of antiholomorphic involutions on* U. *Here* $\mathrm{Aut}^h(U) =$ *biholomorphic automorphisms of* U.

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