BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 5, September 1974

A UNIFIED APPROACH TO GENERALIZED INVERSES OF LINEAR OPERATORS: II. EXTREMAL AND PROXIMAL PROPERTIES

BY M. Z. NASHED AND G. F. VOTRUBA

Communicated by Robert Bartle, December 12, 1973

1. **Introduction.** This note is a sequel to [5]; the notations are the same. An important property of the Moore-Penrose inverse $T^\dagger$ of a matrix, or a bounded linear operator, or a densely-defined closed linear operator T on a Hilbert space is the relation of $T^\dagger$ to extremal solutions of the equation $Tu=y$ (see [2], [4], [7]). We develop proximal properties of generalized inverses in normed spaces within the general setting of [5] and demonstrate their relations to extremal, minimal, and best approximate solutions.

2. **Preliminaries.** Let N be a normed linear space. Let $\operatorname{sp} A$ denote the span of a set A and $d(x, A)$ the distance from x to $A \subset N$. The following definition of *orthogonality* is used: $x \perp y$ means $d(x, \operatorname{sp}\{y\}) = \|x\|$, and $B \perp A$ means $d(x, A) = \|x\|$ for each $x \in B$. Note that this orthogonality is not a symmetric relation. Let M be a subspace of N which has a topological complement, and consider the affine manifold $\mathscr{P}_M = \{P \in \mathscr{L}(N)$: $P^2 = P$ and $\mathscr{R}(P) = M\}$.

PROPOSITION 1. *Let $P_0 \in \mathscr{P}_M$. The following statements are equivalent*:
(a) *P_0 is the nearest point in $\mathscr{P}_M$ to I, and $\|I - P_0\| = 1$.*
(b) $\mathscr{N}(P_0) \perp \mathscr{R}(P_0)$.
(c) *For each $x \in N$, $P_0 x$ is the nearest point in M to x.*

If there exists a $P_0 \in \mathscr{P}_M$ such that any (and hence all) of the statements in Proposition 1 hold, we say that M is an *orthogonally-complemented* subspace of N. We emphasize that if $M = \mathscr{R}(P_0)$ is orthogonally complemented by $\mathscr{N}(P_0)$, it is not necessarily true that $\mathscr{N}(P_0)$ is orthogonally complemented by $\mathscr{R}(P_0)$. Also, orthogonal complements are not necessarily unique. Hilbert spaces are an exceptional case; if M is a closed

---

AMS (MOS) subject classifications (1970). Primary 47A50, 47A99, 41A50; Secondary 54C60, 54C65.

Key words and phrases. Extremal solutions, metric generalized inverse, linear operators, orthogonally-complemented normed subspaces, orthogonal partial inverses, metric selections.