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INDUCTIVELY DEFINED SETS OF REALS

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1. **Introduction.** The concept of inductive definability has become of great interest to recursion theorists in recent years. Recursion over natural numbers, ordinals, and higher type objects may itself be defined by an inductive operator—see for example [7] and [9]. Many results have been obtained characterizing the closures of inductive operators over the natural numbers, and relating lengths of inductive definitions to various interesting ordinals; see [3] for a brief summary.

The purpose of this note is to present results on the closure ordinals and sets of inductive operators over the continuum. Details will appear later in [2], [4], and [5].

2. **Basic definitions and notation.** An inductive operator Γ over a set X is a map from $P(X)$ to $P(X)$ such that for all A, $A \subseteq \Gamma(A)$. Γ determines a transfinite sequence $\{\Gamma^\sigma : \sigma \in \mathrm{ORD}\}$, where $\Gamma^\sigma = U\{\Gamma^\tau : \tau < \sigma\}$ for $\sigma = 0$ or σ a limit and $\Gamma^{\sigma+1} = \Gamma(\Gamma^\sigma)$. Γ is monotone if, for all A, B in $P(X)$, $A \subseteq B$ implies $\Gamma(A) \subseteq \Gamma(B)$. Γ is positive if its application to a set A involves only the positive part of χ_A (the characteristic function of A).

The closure ordinal $|\Gamma|$ of Γ is the least ordinal σ such that $\Gamma^{\sigma+1} = \Gamma^\sigma$; clearly $|\Gamma|$ always has cardinality less than or equal to card(X). The closure $\bar{\Gamma}$ of Γ is $\Gamma^{|\Gamma|}$, the set inductively defined by Γ.

For a class C of inductive operators, the closure ordinal $|C| = \sup\{|\Gamma| : \Gamma \in C\}$ and the closure algebra $\bar{C} = \{A : A$ is 1-1 reducible to $\bar{\Gamma}$ for some Γ in $C\}$. We write C-mon for the class of monotone operators in C.

In studying inductive operators over the continuum, we follow the usual convention that a real number is a function from the set ω of natural numbers to itself; thus the real line is ${}^\omega\omega$.

3. **Main results.** The central result of our research is the following theorem.

THEOREM 1. (a) $|\Pi_1^0\text{-mon}| = |\Pi_1^1\text{-mon}| = |\Sigma_1^1\text{-mon}| = \aleph_1$;
(b) $(\Pi_1^0\text{-mon})^- = (\Pi_1^1\text{-mon})^- = \Pi_1^1$;
(c) $(\Sigma_1^1\text{-mon})^- = \Sigma_2^1$.

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