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ALL OPERATORS ON A HILBERT SPACE ARE BOUNDED

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Introduction. Following Solovay [2], let 'ZF' denote the axiomatic set theory of Zermelo-Fraenkel and let 'ZF + DC' denote the system obtained by adjoining a weakened form of the axiom of choice, DC, (see p. 52 of [2] for a formal statement of DC). From DC a 'countable' form of the axiom of choice is obtainable. More precisely, if $\{B_n : n \in N\}$ is a countable collection of nonempty sets then it follows from DC that there exists a function f with domain N such that $f(n) \in B_n$ for each n.

The system ZF + DC is important because all the positive results of elementary measure theory and most of the basic results of elementary functional analysis, except for the Hahn-Banach theorem and other such consequences of the axiom of choice, are provable in ZF + DC. In particular, the Baire category theorem for complete metric spaces and the closed graph theorem for operators between Fréchet spaces are provable in ZF + DC.

Solovay shows [2] that the proposition, *Each subset of the real numbers is Lebesgue measurable*, cannot be disproved in ZF + DC. He does this by constructing a model for ZF + DC in which the proposition becomes a true statement.

We shall see that the proposition, *Each linear operator on a Hilbert space is a bounded linear operator*, is consistent with the axioms of ZF + DC. Other results of this type are obtained. For example, *Whenever X and Y are separable Fréchet groups and $h: X \to Y$ is a homomorphism then h is continuous*, cannot be proved or disproved in ZF + DC.

Fortunately all the hard work in model theory has been done by Solovay. All that we use here is straightforward functional analysis.

All operators on a Hilbert space are bounded. We recall that a subset S of a topological space T is said to have the *Baire property* if there exists an open set U such that $(U \setminus S) \cup (S \setminus U)$ is meagre. Let BP be the proposition: *Each subset of a complete separable metric space has the Baire property*. In [2, §4], Solovay outlines an argument which shows that when BP is interpreted in his model for ZF + DC then it becomes a true statement. Hence BP is consistent with the axioms of ZF + DC provided Solovay's model exists. We adjoin BP as an axiom and denote the extended system by 'ZF + DC + BP'.

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