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ON FOURIER COEFFICIENTS OF SIEGEL MODULAR FORMS OF DEGREE TWO

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ABSTRACT. Siegel modular forms f of degree two are considered which satisfy: (1) the Fourier coefficients $b_f(R)$, for R a positive definite, semi-integral, primitive matrix, are solely a function of $\det(R)$; and (2) f is an eigenform for the Hecke algebra whose eigenvalues satisfy certain relationships. For such forms, results about multiplicative relationships and asymptotic growth are given, and formulae are given for $b_f(R)$ with R arbitrary in terms of $b_f(T)$ with $\det(2T)$ square-free.

Hecke operators play a vital role in investigating multiplicative relations among Fourier coefficients of modular forms of one complex variable. In this paper, we show that, for a certain class of Siegel modular forms of degree two, Hecke operators play a similar role in determining relations among Fourier coefficients.

Let $f(Z)$ be a Siegel modular form of degree two and weight w. Then $f(Z)$ has a Fourier expansion of the form $f(Z) = \sum_{R \geqq 0} b_f(R)e(RZ)$, where Z is a point in the Siegel upper half plane of degree two, R runs through all positive semidefinite, semi-integral 2×2 matrices, and $e(RZ) = \exp[2\pi i \cdot \mathrm{Trace}(RZ)]$. If $R = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$, with a, c, $2b$ integers, then we set $\gcd(R) = \gcd(a, c, 2b)$. We will denote the determinant of a matrix A by $|A|$.

We now define the Hecke operators (degree two) on the space $\mathscr{F}_w$ of all Siegel modular forms of degree two and weight w. Let

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \qquad \mathscr{L}(n) = \{M \in GL(4, \mathbf{Z}) \colon M^t JM = nJ\}.$$

For f in $\mathscr{F}_w$, n a positive integer, and M in $\mathscr{L}(n)$, we write M in blocks of 2×2 matrices as $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and define

$$(f|_M)(Z) = |M|^{w/2}\, |CZ + D|^{-w} f[(AZ + B)(CZ + D)^{-1}].$$

Noting that one can write $\mathscr{L}(n) = \bigcup_A \mathscr{L}(1)A$, a finite, disjoint union, we define the unnormalized Hecke operator $T(n) \colon \mathscr{F}_w \to \mathscr{F}_w$ as $f|_{T(n)} = \sum_A f|_A$.

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