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EUCLID'S ALGORITHM IN GLOBAL FIELDS

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1. **Introduction.** The purpose of this note is to announce some results regarding the relationship between principal ideal domains and euclidean domains which are subrings of global fields.

Let A be an integral domain. We shall say that A is a euclidean ring, or simply "A is euclidean", if there exists a map $\varphi : A - \{0\} \to N$, N the nonnegative integers, satisfying the following two properties:

(1) If $a, b \in A - \{0\}$, then $\varphi(ab) \geqq \varphi(a)$.

(2) If $a, b \in A$, $b \neq 0$, then there exists $q, r \in A$ such that $a = bq + r$, where $r = 0$ or $\varphi(r) < \varphi(b)$.

It is easy to see that condition (1) is an unnecessary restriction; i.e. if there is a map $\varphi : A - \{0\} \to N$ satisfying only condition (2), then there is always another map φ', derived from φ, such that φ' satisfies both (1) and (2). Further, it is apparently unknown whether one enlarges the class of euclidean integral domains by enlarging N to a well-ordered set of arbitrary cardinality, but this question will not concern us here except to say that whenever A has finite residue classes, i.e., A modulo any nonzero ideal is finite, then insisting on N as a set of values is no restriction. We refer the reader to an excellent paper by P. Samuel [7] in which all of the above and much more is exposed with great clarity.

Let A be as above. We define subsets A_n of A for $n \in N$ by induction as follows: $A_0 = \{0\}$ and if $n \geqq 1$, then $A'_n = \bigcup_{\alpha < n} A_\alpha$. Finally $A_n = \{b \in A |$ there is a representative in A'_n of every residue class of A modulo $bA\}$. Setting $A' = \bigcup_{n \in N} A_n$, A is euclidean if and only if $A' = A$ (see Motzkin [4]). Further when $A' = A$ we get a map $\varphi : A - \{0\} \to N$, where if $x \in A - \{0\}$ then there exists a unique $n \geqq 0$ such that $x \in A_{n+1} - A_n$ and $\varphi(x) = n$. Now not only does φ satisfy conditions (1) and (2) above, but if φ' is any other map satisfying condition (2), then $\varphi(x) \leqq \varphi'(x)$ for all $x \in A - \{0\}$. Hence Motzkin justifiably calls φ the minimal algorithm for A.

Let F be a global field; F is a finite extension of the rational numbers Q, or F is a function field of one variable over a finite field. Let S be a nonempty finite set of prime divisors of F such that S contains all infinite (i.e. archimedean) prime divisors. For each finite (i.e. nonarchimedean) prime divisor P we denote by O_P the valuation ring associated to P in F. Letting

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