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# EIGENFUNCTION EXPANSIONS FOR NONDENSELY DEFINED OPERATORS GENERATED BY SYMMETRIC ORDINARY DIFFERENTIAL EXPRESSIONS[1]

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1. **Nondensely defined symmetric ordinary differential operators.** This note is a sequel to [2]; the notations are the same. Let $L$ be the formally symmetric ordinary differential operator

$$L = \sum_{k=0}^{n} p_k D^k = \sum_{k=0}^{n} (-1)^k D^k \bar{p}_k, \qquad D = \frac{d}{dx},$$

where the $p_k$ are complex-valued functions of class $C^k$ on an interval $a < x < b$, and $p_n(x) \neq 0$ there. In the Hilbert space $\mathfrak{H} = \mathfrak{L}^2(a, b)$ let $S_0$ be the closure in $\mathfrak{H}^2$ of the set of all $\{f, Lf\}$ for $f \in C_0^\infty(a, b)$, the functions in $C^\infty(a, b)$ vanishing outside compact subintervals of $a < x < b$. This $S_0$ in a closed densely defined symmetric operator whose adjoint has the domain $\mathfrak{D}(S_0^*)$, the set of all $f \in C^{n-1}(a, b)$ such that $f^{(n-1)}$ is absolutely continuous on each compact subinterval and $Lf \in \mathfrak{H}$. For $f \in \mathfrak{D}(S_0^*)$, $S_0^* f = Lf$. If $M_0 = S_0^* \ominus S_0$, then

$$\dim(M_0)^{\pm} = \dim \mathfrak{D}((M_0)^{\pm}) = \dim \nu(S_0^* \mp iI) = \omega^{\pm},$$

say ($\nu(T)$ = null space of $T$). Thus $0 \leqq \omega^{\pm} \leqq n$, and $\dim M_0 = \omega^+ + \omega^- \leqq 2n$. Let $\mathfrak{H}_0$ be a subspace of $\mathfrak{H}$, $\dim \mathfrak{H}_0 = p < \infty$, and define the operator $S$, with $\mathfrak{D}(S) = \mathfrak{D}(S_0) \cap (\mathfrak{H} \ominus \mathfrak{H}_0)$, via $S \subset S_0$. We see that (2.1) of [2] is satisfied and Theorem 1 of [2] is applicable to $S$. If $\omega^+ = \omega^- = \omega$, which we now *assume*, then Theorem 2 of [2] is also applicable. For $u, v \in \mathfrak{D}(S_0^*)$ we have Green's formula

$$\int_y^x (\bar{v} Lu - u \overline{Lv}) = [uv](x) - [uv](y),$$

where $[uv]$ is a semibilinear form in $u, u', \ldots, u^{(n-1)}$ and $v, v', \ldots, v^{(n-1)}$. From this it follows that $[uv](x)$ tends to limits $[uv](a)$, $[uv](b)$ as $x$ tends to $a, b$. Then we may write

$$\langle uv \rangle = (Lu, v) - (u, Lv) = [uv](b) - [uv](a).$$

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