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AN INTEGRAL RELATED TO NUMERICAL INTEGRATION

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For real- or complex-valued functions defined on a finite real interval, the concept of integral that is most suitable for numerical approximation is that of Riemann. There the integral is defined as a limit of Riemann sums, any of which can be effectively calculated (given the calculability of the integrand). In fact most common quadrature rules—such as the trapezoid rule, Simpson's rule, or the Gauss-Legendre formulas—do in fact approximate the integral by calculating carefully chosen Riemann sums. Each of these rules converges for the full class of (properly) Riemann integrable functions; there seems to be no larger interesting class of bounded functions for which any quadrature rules converge.

For infinite intervals the situation is not so neat. The improper Riemann integral over $[0, \infty)$ is not defined as a limit of finite sums, and indeed there is no sequence of quadrature formulas

$$Q_n(f) = \sum_{r=1}^{n} a_{r,n} f(x_{r,n})$$

having the property that $Q_n(f) \to \int_0^\infty f$ whenever f is improperly Riemann integrable.[1]

What can we hope for? If we wish to exhibit $\int_0^\infty f$ as a limit of Riemann sums, clearly those sums must be based on partitions of intervals that expand to fill $[0, \infty)$. Furthermore the gauges of those partitions—the lengths of their longest subintervals—must simultaneously go to zero; otherwise we would not get the correct integral even for functions that are zero outside a finite interval.

DEFINITION. A complex-valued function f, defined on $[0, \infty)$, will be called "simply integrable" if there is a number I with the following property: For every $\varepsilon > 0$ there are numbers $B = B(\varepsilon)$ and $\Delta = \Delta(\varepsilon)$ such that if $b > B$ and $\Pi: 0 = x_0 < x_1 < \cdots < x_n = b$ is any partition of $[0, b]$ with $\max\{(x_r - x_{r-1})\} < \Delta$ and $\xi_1, \xi_2, \ldots, \xi_n$ are any numbers satisfying

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[1] In this paper the improper Riemann integral $\int_0^\infty f(x)dx$ is understood as the finite limit of the proper Riemann integral $\int_0^\infty f(x)dx$ as $b \to \infty$.