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NONCOMMUTATIVE LOCALIZATION

BY JOACHIM LAMBEK

0. **Introduction.** To motivate this exposition, let us begin by looking at an example. If $\boldsymbol{Z}$ is the ring of integers and p is a prime number, we have embeddings $\boldsymbol{Z} \to \boldsymbol{Z}_p \to \hat{\boldsymbol{Z}}_p$. Here the first arrow indicates "localization at p", $\boldsymbol{Z}_p$ being the local ring of quotients at p. We may take this to be the ring of rationals with denominators prime to p, described explicitly as a direct limit

$$\boldsymbol{Z}_p = \varinjlim\{\mathrm{Hom}((n), \boldsymbol{Z}) \mid p \nmid n\}$$

where $(n) = n\boldsymbol{Z}$ is the principal ideal generated by n, n running over all positive integers not divisible by p. The second arrow indicates "p-adic completion". $\hat{\boldsymbol{Z}}_p$ is the completion of $\boldsymbol{Z}_p$ (incidentally also of $\boldsymbol{Z}$) in the p-adic topology. It is also known as the ring of p-adic integers and may be described explicitly as an inverse limit

$$\hat{\boldsymbol{Z}}_p = \varprojlim\{\boldsymbol{Z}_p/(p^k) \mid k \geqq 0\},$$

where $(p^k) = p^k\boldsymbol{Z}_p$ is the principal ideal generated by p^k, k running over all natural numbers.

In commutative algebra one is always told first to localize and then to complete. It is therefore not surprising that the combination of these two processes can be described in a simpler fashion than either. In fact, $\hat{\boldsymbol{Z}}_p$ is the ring of endomorphisms of the Prüfer group

$$\boldsymbol{Z}/(p^\infty) = \varinjlim\{\boldsymbol{Z}/(p^k) \mid k \geqq 0\}.$$

The latter may also be regarded as the group of all rationals with denominators powers of p modulo $\boldsymbol{Z}$. More significantly, it is the injective hull of $\boldsymbol{Z}/(p)$.

We want to generalize this result in two directions. First, instead of $\boldsymbol{Z}$

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