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BOUNDARY BEHAVIOR OF THE CARATHÉODORY, KOBAYASHI, AND BERGMAN METRICS ON STRONGLY PSEUDOCONVEX DOMAINS IN C^n WITH SMOOTH BOUNDARY

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Let G be a bounded domain in $\boldsymbol{C}^n$. Let Δ be the unit disk in $\boldsymbol{C}$. Let $\Delta(G)$ be the set of holomorphic mappings from G to Δ, and $G(\Delta)$ the set of holomorphic mappings from Δ to G. The Carathéodory metric on G (i.e. the infinitesimal form, as in [7] is defined by

$$F_C(z, \xi) = \sup_{f \in \Delta(G)} |f_*(\xi)| = \sup_{f \in \Delta(G)} \left| \sum_{i=1}^n \frac{\partial f}{\partial z_i}(z)\xi_i \right|, \qquad z \in G, \xi \in \boldsymbol{C}^n.$$

The Kobayashi metric on G (infinitesimal form) is defined by [**8**] $F_K(z, \xi) = \inf\{\alpha | \exists f \in G(\Delta) \text{ with } f(0) = z,\ f'(0) = \xi/\alpha,\ \alpha > 0\}$. For the definition of the Bergman metric see [**1**] or [**4**]. We take

$$F_B(z, \xi) = (ds^2(z, \xi))^{\frac{1}{2}}$$

in the notation of [**4**].

We consider the boundary behavior of these metrics for fixed ξ. The notable features are (i) the different limiting behavior in tangential and normal directions (cf. Stein [**9**]), and (ii) the appearance of the Levi form as the limiting value of a quantity defined inside the domain.

THEOREM. *Let G be a (bounded) strongly pseudoconvex domain in $\boldsymbol{C}^n$ with C^2 boundary. Let $z_0 \in \partial G$. Let φ be a C^2 defining function for ∂G such that $|\nabla_z \varphi(z_0)| = 1$. Let $F(z, \xi)$ be either the Carathéodory or the Kobayashi metric on G. Then*

$$\lim_{z \to z_0} F(z, \xi) d(z, \partial G) = \tfrac{1}{2}|\nabla_z \varphi(z_0) \cdot \xi| = \tfrac{1}{2}|\xi_N(z_0)|.$$

If $\nabla_z \varphi(z_0) \cdot \xi = 0$, i.e. ξ is a tangent vector to ∂G at z_0, then

$$\lim_{z \to z_0; z \in \Lambda} (F(z, \xi))^2 d(z, \partial G) = \tfrac{1}{2}\mathscr{L}_{\varphi, z_0}(\xi) = \tfrac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(z_0)\xi_i \bar{\xi}_j.$$

$d(z, \partial G)$ is the Euclidean distance to the boundary. $\nabla_z \varphi$ is the vector $(\partial\varphi/\partial z_1, \ldots, \partial\varphi/\partial z_n)$, and $\nabla_z \varphi(z_0) \cdot \xi = \sum_{i=1}^n (\partial\varphi/\partial z_i)(z_0)\xi_i = \xi_N(z_0)$ is the (complex) normal component of ξ at z_0. Λ in the second limit denotes a cone of arbitrary aperture with vertex at z_0 and axis the interior normal to ∂G.

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