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Φ-LIKE ANALYTIC FUNCTIONS. I

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The object of this paper is to introduce a very broad generalization, indeed a complete generalization of star-like and spiral-like functions. Our principal definition is the following.

DEFINITION 1. Let f be analytic in the unit disk $\Delta = \{z : |z| < 1\}$ of the complex plane with $f(0) = 0$, $f'(0) \neq 0$. Let Φ be analytic on $f(\Delta)$ with $\Phi(0) = 0$, $\operatorname{Re} \Phi'(0) > 0$. Then *$f$ is Φ-like (in Δ)* if

$$\text{(1)} \qquad \operatorname{Re}(zf'(z)/\Phi(f(z))) > 0 \qquad (z \in \Delta).$$

REMARKS. 1. The two classical cases of Definition (1) are given by $\Phi(w) = w$ (f is star-like) and, more generally, $\Phi(w) = \lambda w$, $\operatorname{Re} \lambda > 0$. ($f$ is spiral-like of type $\arg \lambda$.)

2. The conditions $\Phi(0) = 0$, $\operatorname{Re} \Phi'(0) > 0$ on Φ are necessary for the existence of an f as described satisfying (1). Conversely, if Φ, analytic in a neighborhood of 0, has these two properties, then there exist Φ-like functions f. For the present we mention only the trivial example $f(z) = az$, where $|a|$ is nonzero and sufficiently small.

3. In spite of the great generality of Definition 1, Φ-like functions are necessarily univalent in Δ (Theorem 1). Moreover the converse is true: Every function analytic and univalent in Δ and vanishing at 0 is Φ-like for some Φ (Corollary 1). Thus we shall obtain a characterization of univalence.

4. The definition immediately below will prove to be the geometric counterpart of Definition 1. (See Theorems 1 and 2.)

DEFINITION 2. Let Ω be a region containing 0, and let Φ be analytic on Ω with $\Phi(0) = 0$, $\operatorname{Re} \Phi'(0) > 0$. Then *$\Omega$ is Φ-like* if for any $\alpha \in \Omega$ the initial value problem

$$\text{(2)} \qquad dw/dt = -\Phi(w), \qquad w(0) = \alpha$$

has a solution $w(t)$ defined for all $t \geqq 0$ such that $w(t) \in \Omega$ for all $t \geqq 0$ and $w(t) \to 0$ as $t \to +\infty$.

REMARKS. 5. If there is a solution of (2) on $[0, \infty)$, it is necessarily unique by a fundamental theorem on first order differential equations. For instance if $\alpha = 0$, then $w(t) = 0$ for all t.

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