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THE HOPF BIFURCATION FOR NONLINEAR SEMIGROUPS

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1. **Introduction.** Several authors, e.g. [2], [9], [10], [12], [17], have shown by perturbation techniques that the Hopf theorem (see [8], [16]) on the development of periodic stable solutions is valid for the Navier-Stokes equations; in particular, solutions near the stable periodic ones remain defined and smooth for all $t \geqq 0$. The principal difficulty is that the Hopf theorem deals with flows of smooth vector fields on finite-dimensional spaces, whereas the Navier-Stokes equations define a flow (or evolution operator) for a nonlinear partial differential operator (actually it is a nonlocal operator).

The aim of this note is to outline a method for overcoming this difficulty which is entirely different in appearance from the perturbation approach. The method depends on invariant manifold theory [6], [7] plus certain smoothness properties of the flow which actually hold for the Navier-Stokes flow.

While the statements of the results are relatively simple, the proofs are somewhat complicated in their details. They will not be presented in full here, because all the details of the relevant invariant manifold theory have not yet been published (several details on the applications became clear only after conversations with M. Hirsch). We expect to write out these proofs at a later date. Nevertheless, the summary presented here should give a picture of the method and some idea of the proofs. We hope that the nature of the hypotheses allows our formulation to be fairly readily applicable. There are some indications that this is so. (This is based on conversations with N. Kopell who is working on the bifurcation of periodic solutions in certain chemical systems.)

We shall be considering a system of evolution equations of the general form

$$dx/dt = Y(x), \qquad x(0) \text{ given},$$

where Y is an operator on a suitable function space $\mathscr{H}$ and will eventually depend on a parameter μ. For example, Y may be the Navier-Stokes operator and μ the Reynolds number. This system is assumed to define unique local solutions $x(t)$ and thereby a flow F_t which maps $x(0)$ to $x(t)$.

The key thing we need to know about the flow F_t of our system is that, for each *fixed t*, F_t is a C^∞ mapping on a suitable Hilbert space $\mathscr{H}$ (F_t is

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