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ON THE NONEXISTENCE OF SOLUTIONS OF DIFFERENTIAL EQUATIONS IN NONREFLEXIVE SPACES[1]

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We consider the problem of the existence of solutions of ordinary differential equations in Banach spaces. Counterexamples in c_0 by Dieudonné [2] and in l_2 by Yorke [4] show that, in infinite-dimensional spaces, Peano's existence theorem need not necessarily be true. A natural question then is that of asking whether there could exist infinite-dimensional Banach spaces on which Peano's theorem holds, or otherwise, whether the truth of Peano's theorem is a characterization of the finite dimensionality.

This paper offers only a partial answer to this question: Our Theorem 1 below states that there exist no nonreflexive spaces on which Peano's theorem holds.

THEOREM 1. *Let X be a nonreflexive Banach space. Then there exists a continuous $F: R \times X \to X$ such that the Cauchy problem*

$$\text{(CP)} \qquad x' = F(t, x), \qquad x(0) = 0,$$

admits no solution on any nonvanishing interval $[a, b]$ containing the origin.

PROOF. Call B the unit ball of X. From R. C. James' characterization of reflexivity [3] it follows that there exists a $v \in X^*$, $\|v\| = 1$, such that, for every $x \in B$, $\langle v, x\rangle < 1$. We shall now construct a fixed-point-free continuous mapping f of B into itself with some special properties.

By definition of norm of v, we can recursively define a sequence of points x_n, $\|x_n\| = 1$, such that $\langle v, x_n\rangle < \langle v, x_{n+1}\rangle$ and $\langle v, x_n\rangle \to 1$, and consider the sets

$$O_1 = \{x \in B : \langle v, x\rangle < 2\langle v, x_2\rangle - 1\},$$

$$O_n = \{x \in B : 2\langle v, x_{n-1}\rangle - 1 < \langle v, x\rangle < 2\langle v, x_{n+1}\rangle - 1\}.$$

Then every O_j is open (relative to B) and their union covers B. Moreover it is not difficult to check that every point $x \in B$ belongs to at most two O_j and has a neighborhood that meets at most three O_j, so that the covering $\{O_n\}$ is locally finite and has a partition of unity subordinated

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