BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 6, November 1972

# A NECESSARY AND SUFFICIENT CONDITION FOR THE CONVERGENCE OF A SEQUENCE OF ITERATES FOR QUASI-NONEXPANSIVE MAPPINGS

BY W. V. PETRYSHYN[1] AND T. E. WILLIAMSON, JR.

Communicated by Eugene Isaacson, May 1, 1972

**Introduction.** The purpose of this note is to present two theorems which provide necessary and sufficient conditions for the convergence of the successive approximation method (Theorem 1) and of the convex combination iteration method (Theorem 2) for quasi-nonexpansive mappings defined on suitable subsets of Banach spaces and with nonempty sets of fixed points. We also indicate briefly how these theorems are used to deduce a number of known, as well as some new, convergence results for various special classes of mappings of nonexpansive, $P$-compact, and 1-set-contractive type which recently have been extensively studied by a number of authors. Complete proofs and detailed discussion of the results presented in this note will be given in [**15**].

1. Let $X$ be a real Banach space, $D$ a closed subset of $X$, and $T$ a continuous mapping of $D$ into $X$ such that $T$ has a nonempty set of fixed points $F(T) \subset D$ and

(1) $\|T(x) - p\| \leqq \|x - p\|$ for all $x$ in $D$ and $p$ in $F(T)$.

In what follows we shall refer to $T$ satisfying the above conditions as *quasi-nonexpansive*. Condition (1) was introduced by Tricomi [**17**] for real functions and further studied by Diaz and Metcalf [**3**] and Dotson [**4**] for mappings in Banach spaces. It is not hard to see that the class of quasi-nonexpansive mappings properly includes the class of nonexpansive maps (i.e., $T$ is such that $\|Tx - Ty\| \leqq \|x - y\|$ for $x, y \in D$) with $F(T) \neq \varnothing$.

The first basic result of this note is the following new theorem which characterizes the convergence of Picard iterates for quasi-nonexpansive maps.

THEOREM 1. *Let $X$ be a real Banach space, $D$ a closed subset of $X$, and $T$ a quasi-nonexpansive mapping of $D$ into $X$. Suppose there exists a*

*AMS* 1970 *subject classifications*. Primary 47H15; Secondary 47H05.
*Key words and phrases*. Iteration methods, convergence, quasi-nonexpansive mapping, fixed points, nonexpansive mappings, condensing and 1-set-contractive mappings, $P$-compact mappings.
[1] Partially supported by the NSF grant GP-20228.