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PRIME NUMBER THEOREMS FOR THE COEFFICIENTS OF MODULAR FORMS

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1. **Introduction.** The Ramanujan function $\tau(n)$ is defined as the nth coefficient in the q-expansion of the discriminant function

$$(1) \qquad \Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n, \qquad q = e^{2\pi i z}, \operatorname{Im}(z) > 0.$$

It is well known that $\Delta(z)$ spans the space of cusp forms of dimension -12 associated with the unimodular group and is in fact an eigenfunction of the Hecke operators. With $\Delta(z)$ there is associated a Dirichlet series with an Euler product of the type

$$(2) \qquad \varphi(s) = \sum_{n=1}^{\infty} \tau(n) n^{-s} = \prod_{p} (1 - \tau(p) p^{-s} + p^{11-2s})^{-1}.$$

The Dirichlet series $\varphi(s)$ defines a regular function in $\operatorname{Re}(s) > 13/2$ and satisfies a functional equation of known type. Hardy [3] was the first to observe that the location of the zeros of $\varphi(s)$ gives rise to problems similar to those for the Riemann zeta function $\zeta(s)$. In the Princeton version of [3] Hardy suggested that a prime number theorem for the Ramanujan function

$$(3) \qquad \sum_{p \leqq x} \tau(p) \log p = O(x^{13/2})$$

would follow if one could show that $\varphi(s) \neq 0$ for $\operatorname{Re}(s) = 13/2$. In [6], Rankin settled Hardy's problem by showing that indeed $\varphi(s)$ does not vanish on $\operatorname{Re}(s) = 13/2$. It is also implicit in Rankin's work that the large O in (3) can be replaced by small o. In this note we indicate further improvements on Rankin's result.

2. **Statement of results.** As in the classical situation of the Riemann zeta function, the following results lead to an improvement of (3).

LEMMA 1 (GENERALIZED VON MANGOLDT FORMULA). *Let $N_\varphi(T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\varphi(s)$ with $11/2 \leqq \beta \leqq 13/2$ and $0 \leqq \gamma \leqq T$.*

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