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INDEPENDENCE OF THE PRIME IDEAL THEOREM FROM THE HAHN BANACH THEOREM

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1. **Introduction.** Principles in diverse areas of mathematics have been proven equivalent to the axiom of choice, AC. The same can be said of the prime ideal theorem for Boolean algebras, PI. AC is independent of PI by [2]. The Hahn Banach theorem, HB, is a consequence of PI [3], [7] and also has some surprising equivalent forms [4].

THEOREM 1. *PI is independent of HB in (the usual) ZF set theory.*

The Krein Milman theorem, KM, is a consequence of AC and PI + KM → AC.[1] This, together with the result of [1], shows HB + VKM → AC where VKM is a strengthened version of KM. Let ZFA be the weakening of ZF to permit the existence of a set of atoms.

THEOREM 2. *AC is independent of HB + KM in ZFA.*

It is open whether ZF can replace ZFA in Theorem 2. We are grateful to W.A.J. Luxemburg who helped state these results in their present form.

2. **The model with atoms.** We use the original permutation model of Fraenkel. This model, and variants of it, is discussed by Mostowski in [5]. We do not give the construction of the model here. Instead we list some statements which are true in the model and base our subsequent arguments on these statements. The statements follow easily from observations of [5] and [6].

1. The axioms of ZFA hold.
2. There is a function P on ω such that the P_i are mutually disjoint pairs and $\mathrm{K} = \bigcup_{i \in \omega} P_i$ is the set of atoms.
3. There is a relation $\nabla(n, x)$, also written $x \in \nabla_n$, which satisfies:

(a) Every set is in some ∇_n, $n \in \omega$.

(b) Each ∇_n contains all ordinals, the function P, and the members of $\bigcup_{i<n} P_i$.

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[1] This result has been independently proved by Peter Renz, W.A.J. Luxemburg, and (jointly) J.L. Bell, and D.H. Fremlen. Peter Renz communicated the result to us.