BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 6, November 1971

LEFSCHETZ FORMULA AND MORSE INEQUALITIES

BY EVERETT PITCHER

Communicated May 24, 1971

The Lefschetz fixed point formula, when applied to a gradient vector field, is the same equality as the last of the Morse inequalities of critical point theory. The latter, when applicable, contain more information than the Lefschetz fixed point formula. The Lefschetz fixed point formula has an essentially wider range of applicability, for example to maps not homotopic to the identity and hence not appearing in a deformation theory of critical points. One can ask either how to obtain more information to accompany the Lefschetz fixed point formula or how to achieve wider applicability for a theory like critical point theory.

We note that Smale has extended critical point theory to a class of vector fields which are not gradient fields and states that a similar theory applies to a class of diffeomorphisms not necessarily homotopic to the identity. (See [S].) Our approach is different in that our maps need be neither homotopic to the identity nor diffeomorphisms.

The prodecure is the following. The idea of the characteristic polynomial of an endomorphism of a finitely generated abelian group has been extended to that of a characteristic rational function. See references [F], [K-S], and [M] for both accounts of the ideas and earlier references to Fuller and Weil. The characteristic rational function is one of a set of invariants to be defined. The application is to downward maps, defined below, which do not necessarily have the property of being homotopic to the identity. The relations in Theorem 2 generalize the Morse inequalities and reduce to them in the subclass of downward maps homotopic to the identity. The relations in Theorem 2 incorporate the Lefschetz fixed point formula for the class of downward maps in a set of inequalities. Theorem 2 follows from Theorem 1, which contains more information in the coefficients other then the traces. Although the development can be given an abstract formulation in spectral homology theory, we shall confine discussion here to a concrete formulation sufficient for the application.

Suppose that G is an abelian group finitely generated over the integers J and that $f:G\to G$ is an endomorphism. Let $\phi_f(\lambda)$ denote the characteristic polynomial, a monic polynomial whose degree n is the rank of G.

AMS 1970 *subject classifications*. Primary 57D70, 55C20; Secondary 58E05.