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# ON THE NONEXISTENCE OF COMPLEX HAAR SYSTEMS[1]

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1. **Introduction.** Schoenberg and Yang [8] have shown that a finite polyhedral set $X$ admits a complex Haar system only if $X$ is embeddable in the plane. We replace the requirement that $X$ be a finite polyhedral set with several weaker assumptions.

Let $X$ be a compact Hausdorff space, and let $C(X)$ be the linear space of continuous complex valued functions on $X$. A subspace $M$ of $C(X)$ of dimension $n \geqq 2$ is said to be a complex Haar system if and only if each nonzero member of $M$ has at most $n-1$ zeros in $X$. Haar and Kolmogoroff (see [6, Theorem 19]) showed that Haar systems are precisely those finite-dimensional subspaces of $C(X)$ that permit a unique best Chebyshev approximation to each $f$ in $C(X)$.

This article owes its being to Professor R. Creighton Buck who supervised its writing in my dissertation [7]. Credit is also due Professor Edward R. Fadell who made many useful suggestions.

2. **Main results.** By a $k$-ode we mean a homeomorph of the subspace of the plane consisting of $k$ distinct radii of unit length drawn from the origin, and by a disk we mean a homeomorph of the closed unit disk. Also, we will say that $X$ is of type $H$ if and only if $X$ is a compact connected Hausdorff space such that $C(X)$ contains a Haar system. Embeddable always means "in the plane."

In my dissertation I showed:

(A) *A space of type H that contains a disk is embeddable*; and

(B) *a locally connected space of type H that contains as an open set a k-ode for some* $k \geqq 3$ *is embeddable.*

Also, I conjectured:

(C) *Any locally connected space of type H is embeddable*; and

(D) *not every space of type H is embeddable.*

Since then, Professors Brian R. Ummel and George Henderson of the University of Wisconsin, Milwaukee, have verified (C).

In summary we now have

THEOREM. *Any space X of type H that is not embeddable is a* 1-

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