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ON MEASURABILITY OVER PRODUCT SPACES[1]

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The main result announced here is a negative solution of the Kakutani-Doob problem [3] on measurability of stochastic processes, assuming the continuum hypothesis. Thus the positive solution proposed earlier by M. Mahowald [6] is incorrect (the last step in the argument applies the Fubini theorem to sets in a product space which need not be measurable).

Complete proofs will appear in the Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability. Preprints are available from the author.

The problem can be formulated as follows. Let

$$x = (x_t(\omega), 0 \leqq t \leqq 1, \omega \in \Omega)$$

be a real-valued stochastic process on $I = [0, 1]$ over some probability space (Ω, P). Then by a theorem of Kolmogorov, x has a probability distribution P_x on the space R^I of all functions from I into the real line R. We embed R in a compact space, such as its one-point compactification $\overline{R}$. Now $\overline{R}^I$ is also a compact Hausdorff space and P_x defines a Baire measure on $\overline{R}^I$. We take the unique regular Borel extension $\overline{P}_x$ of P_x (cf. Kakutani [5], E. Nelson [7]).

Let E be the evaluation map $(t, f) \rightarrow f(t)$ from $I \times \overline{R}^I$ into $\overline{R}$, and let λ be Lebesgue measure on I. Then the process $E(t, f)$ has the same probability laws as the original x, i.e. $P_E = P_x$, where P_E is defined from P_x as P_x from P.

Thus we have a "canonical" representative from each class of processes x with a given P_x. In general, E need not be measurable for $\lambda \times \overline{P}_x$ [5]. The Kakutani-Doob problem asks: if x is $\lambda \times P$-measurable, then is E measurable for $\lambda \times \overline{P}_x$?

A negative answer, assuming the continuum hypothesis, can be given for certain processes of the form

$$x(t, \omega) = \sum y_n(t) z_n(\omega)$$

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