TOEPLITZ OPERATORS IN A QUARTER-PLANE[1]

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The key result about Toeplitz operators on a quarter-plane is apparently this, that the Fredholm alternative holds if the symbol t, in the class of nonvanishing functions on the torus, is homotopic to a constant. We want to give a simple proof.

Recall that the Toeplitz operator T with symbol $t(\theta)=\sum t_k e^{-ik\theta}$, acting on a vector v, is defined by

$$(Tv)_i = \sum_{j\geqq 0} t_{i-j}v_j, \qquad i \geqq 0. \tag{1}$$

In the classical one-dimensional case, i, j, and k are integers, θ is a scalar, and T is represented by the discrete convolution matrix

$$\begin{pmatrix} t_0 & t_1 & t_2 \cdots \\ t_{-1} & t_0 & t_1 \cdots \\ t_{-2} & t_{-1} & t_0 \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}. \tag{2}$$

On the quarter-plane, i and j are *pairs* of nonnegative integers, $\theta=(\theta_1, \theta_2)$, $k\theta=k_1\theta_1+k_2\theta_2$, and T can be represented by an array (2) in which each entry is itself a Toeplitz operator on a half-line—or equivalently, by a four-dimensional analogue of (2). We assume $\sum |t_k| < \infty$, so that T is a bounded operator on any of the spaces l^p_+, normed by

$$\|v\|_p = \Big(\sum_{j\geqq 0} |v_j|^p\Big)^{1/p}.$$

After Fourier transform, taking the vector v into $v(\theta)=\sum v_j e^{ij\theta}$, the action of T remains easy to describe:

$$(Tv)(\theta) = P(t(\theta)v(\theta)), \tag{3}$$

where P is projection onto the nonnegative frequencies,

$$P\sum a_j e^{ij\theta} = \sum_{j\geqq 0} a_j e^{ij\theta}. \tag{4}$$

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