ON THE FREENESS OF ABELIAN GROUPS: A GENERALIZATION OF PONTRYAGIN'S THEOREM[1]

BY PAUL HILL

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Recall that a subgroup H of a torsion-free abelian group G is pure if and only if G/H is again torsion-free. For brevity, call a set S an $f\sigma$-union of its subsets S_λ, $\lambda \in \Lambda$, if each finite subset of S is contained in some S_λ. In this language, Pontryagin's theorem is (equivalent to) the following, where the set-theoretic, not the group-theoretic, union prevails.

THEOREM (PONTRYAGIN). *If the countable, torsion-free abelian group G is the $f\sigma$-union of pure subgroups that are free, then G must be free.*

Pontryagin gave an example that demonstrates that the prefix "$f\sigma$" cannot be deleted from the above theorem; indeed he showed that there exists a torsion-free group of rank 2 that is not free such that each subgroup of rank 1 is free (see, for example, [1, p. 151]). We present the following direct generalization of Pontryagin's theorem obtained by transposing the countability condition.

THEOREM 1. *If the torsion-free abelian group G is the $f\sigma$-union of a countable number of pure subgroups that are free, then G must be free.*

OUTLINE OF PROOF. Let G be an $f\sigma$-union of pure subgroups H_n, $n<\omega$, that are free. Write $H_n = \sum_{i\in I(n)}\{g_i\}$. For simplicity of notation, let μ denote the smallest ordinal having the cardinality of G. We claim that there exist subgroups A_α, $\alpha<\mu$, of G satisfying the following conditions:

(0) $A_0=0$.

(1) A_α is pure in G for each $\alpha<\mu$.

(2) $\{A_\alpha, H_n\}$ is pure in G for each $\alpha<\mu$ and each $n<\omega$.

(3) $A_{\alpha+1}\supseteq A_\alpha$ for each α such that $\alpha+1<\mu$.

(4) $A_{\alpha+1}/A_\alpha$ is countable for each α such that $\alpha+1<\mu$.

(5) $A_\alpha \cap H_n = \sum_{i\in I(n,\alpha)}\{g_i\}$ for $\alpha<\mu$ and $n<\omega$, where $I(n, \alpha)$ is a subset of $I(n)$.

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