BEST UNIFORM APPROXIMATIONS VIA ANNIHILATING MEASURES

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The problem under consideration in this paper is that of uniformly approximating an arbitrary continuous function g on the closed unit disk $\overline{D}$ by continuous functions f which are analytic in $D=\{z$ complex: $|z|<1\}$. In particular, we are concerned with the existence, uniqueness, and construction of a best approximation f_0 to g. Our results consist of a proof of the uniqueness of f_0 when it exists and an algorithm for constructing f_0 for certain classes of functions g. Both results follow from a more general theorem on best uniform approximations and annihilating measures.

If E is a normed linear space, A is a subspace of E, and S_A^* consists of all the linear functionals L on E with $\|L\|\leqq 1$ and which vanish on A then, as a consequence of the Hahn-Banach theorem, the following relationship holds [1].

THEOREM 1. *If $g\in E$ then*

$$\|g\|_A = \inf_{f\in A}\|g-f\| = \max_{L\in S_A^*} |L(g)|.$$

For $E=C(K)$, the continuous complex valued functions defined on the compact Hausdorff space K, additional information can be obtained from Theorem 1 by applying the Riesz representation theorem [4] to $L\in S_A^*$. Here $\|g\|=\max_{z\in K}|g(z)|$ is the uniform norm.

THEOREM 2. *If $g\in C(K)$, $f_0\in A$ is a best uniform approximation to g, $L\in S_A^*$, and $L(g)=\|g\|_A$ then $g-f_0=\|g\|_A\bar{\phi}$ a.e. $d\mu$ where $\phi d\mu$ is the polar decomposition of the unique regular Borel measure on K which represents L.*

PROOF. By Theorem 1, there is an $L\in S_A^*$ with $L(g)=\|g\|_A$ and $\|L\|=1$. Let $\phi d\mu$ be the measure which represents L where $|\phi|=1$ a.e. $d\mu$, $d\mu\geqq 0$ and $\int_K d\mu=1$. Now,

$$\|g\|_A = \int_K (g-f_0)\phi d\mu \leqq \int_K |(g-f_0)\phi|\, d\mu \leqq \int_K \|g-f_0\| d\mu = \|g\|_A.$$

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