CLASS NUMBERS OF POSITIVE DEFINITE QUATERNARY FORMS[1]

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1. Introduction. Let V be a quadratic vector space over the field of rational numbers $\boldsymbol{Q}$. We assume that the associated quadratic form q is positive definite with square discriminant. Let M be a lattice in V which is maximal integral with respect to q. We denote by H the number of proper classes of maximal integral lattices. The purpose of this note is to announce a formula for H. This formula is derived by applying the Selberg Trace Formula in an appropriate manner. The method we employ is motivated by the successful use of the Selberg Trace Formula in the computation of ideal class numbers of quaternion algebras over $\boldsymbol{Q}$ (cf. [6]).

Since q has square discriminant, we may assume that $V=\mathfrak{A}$, a quaternion (division) algebra over $\boldsymbol{Q}$, and $q=N$, the norm form of $\mathfrak{A}$. We may take M to be $\mathfrak{O}$, a fixed maximal order in $\mathfrak{A}$. If x_1, x_2, x_3, x_4 is a basis of $\mathfrak{O}$ over the ring of integers $\boldsymbol{Z}$, then the discriminant of $\mathfrak{O}$ with respect to the norm form N is $=|\mathrm{Tr}(x_i x_j^*)|=D$, the discriminant of the quaternion algebra $\mathfrak{A}$. Here * is the canonical involution of $\mathfrak{A}$. It is well known that $D=d^2$, where d is a positive square-free integer. Let us write $d=p_1\cdots p_e$, where the p_i, $i=1,\cdots,e$ are distinct prime numbers. We recall that $\{p_1,\cdots,p_e\}$ is the set of finite primes p such that $\mathfrak{A}_p=\mathfrak{A}\otimes_{\mathbf{Q}}\boldsymbol{Q}_p$ is a division algebra over $\boldsymbol{Q}_p$, the field of p-adic numbers. One calls $p_1,\cdots,p_e$ the *nonsplit* or *ramified* primes of $\mathfrak{A}$.

We do not apply the Selberg Trace Formula in the setting afforded by the orthogonal groups which appear in the usual definition of H. The reason is that the definitions of these groups involve a norm condition which makes integration unmanageable and which also complicates conjugacy considerations. To avoid these difficulties, we replace the usual definition of H by one which is more suitable for our

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