NONEXPANSIVE RETRACTS OF BANACH SPACES

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In what follows, C is a closed convex subset of the real, reflexive, strictly convex Banach space X. If $F \subset C$, we shall call F a nonexpansive retract of C if either $F = \emptyset$ or there is a retraction of C onto F which is a nonexpansive mapping.

THEOREM 1. *If T: $C \to C$ is nonexpansive, then $F(T)$, the fixed point set of T, is a nonexpansive retract of C.*

THEOREM 2. *The class of nonexpansive retracts of C is closed under arbitrary intersections.*

To prove these theorems, suppose F is a nonempty subset of C, and set $\mathfrak{F} = \{f: C \to C \mid f \text{ is nonexpansive and } F \subset F(f)\}$. Define an order on $\mathfrak{F}$ by setting $f < g$ if $\|fx - fy\| \leqq \|gx - gy\|$ for all $(x, y) \in C \times C$, with strict inequality holding for at least one pair (x, y); then set $f \leqq g$ to mean $f < g$ or $f = g$. Then $\leqq$ is a partial ordering of $\mathfrak{F}$.

Every linearly ordered subset of $\mathfrak{F}$ has a lower bound in $\mathfrak{F}$; the proof of this fact utilizes the local weak compactness of C and the weak lower semicontinuity of the norm. Therefore, by Zorn's lemma, $\mathfrak{F}$ has a minimal element.

The strict convexity of X implies that for each $g \in \mathfrak{F}$ there exists a $g_0 \in \mathfrak{F}$ with $F(g_0) = F(g)$ and such that whenever $\|g_0(u) - g_0(w)\| = \|u - w\|$, then $g_0(u) - g_0(w) = u - w$. For example, we may take $g_0 = \frac{1}{2}I + \frac{1}{2}g$, where I is the identity function for C.

Suppose f is a minimal function in $\mathfrak{F}$, and g is any function of $\mathfrak{F}$. Let g_0 be the function of the preceding paragraph; then $g_0 f \in \mathfrak{F}$ while $g_0 f \leqq f$. By the minimality of f, therefore $g_0 f = f$.

Letting $R(f)$ denote the range of f, therefore

$$F(f) \subset R(f) \subset F(g_0) = F(g), \tag{1}$$

and in particular,

$$F(f) \subset F(g) \qquad \text{for } g \in \mathfrak{F}. \tag{2}$$

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