ON THE SPAN OF A RIEMANN SURFACE

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In 1952 Virtanen [7] gave an example of a Riemann surface which carries nonconstant AD-functions, yet there is a point on the surface where every such function has a critical value, i.e. a vanishing derivative. It had already been pointed out in [1] that this phenomenon cannot occur for planar surfaces. It has remained an open question whether this situation prevails for harmonic functions. This unsolved problem was restated in the monograph *Capacity functions* (Sario-Oikawa [5]). Problem 4 in the list of open questions at the end of that book asks, more generally, whether the vanishing of the span S_m at one point on a Riemann surface implies that it vanishes at all points on the surface.

In this paper we describe a Riemann surface R with the property that for any preassigned positive integer m there is a point on R where every HD-function has a critical point of order m. Yet R carries nonconstant HD-functions. Thus Problem 4 of [5] is solved for the H-span. Nevertheless, there are still unresolved questions in this area when we restrict ourselves to surfaces of finite genus or to KD-functions.

Consider the surface of Tôki [6] (this surface is also constructed in Ahlfors-Sario [2]; see No. 25 of §8, Chapter IV). This surface is realized as a unit disk with infinitely many radial slits identified in a prescribed manner. It is of class $O_{HD}-O_G$, and hence the Royden harmonic boundary [4] consists of exactly one point.

Remove a disk $\{|z|<\epsilon\}$ from Tôki's surface and then form the double across $\{|z|=\epsilon\}$. The resulting surface is realized as a radial slit annulus $\{\epsilon^2<|z|<1\}$ with certain identifications among the edges of the slits. For a preassigned positive integer m, insert an additional $m+1$ radial slits symmetric to the origin and identify their edges cyclically. The resulting surface R has a branch point of order m, and the Royden harmonic boundary of R consists of two points. Therefore the space of HD-functions on R is two-dimensional.

The function $\log r$ can be regarded as an HD-function on R, and in that sense it has critical point of order m at the branch point of order m. The same is true for any HD-function u on R since

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