ON THE EXTENSION OF LIPSCHITZ, LIPSCHITZ-HÖLDER CONTINUOUS, AND MONOTONE FUNCTIONS[1]

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1. **Introduction.** The well-known theorem of Kirszbraun [9], [14] asserts that a Lipschitz function from R^n to itself, with domain a finite point-set, can be extended to a larger domain including any arbitrarily chosen point. (The Euclidean norm is essential; see Schönbeck [16], Grünbaum [8].) This theorem was rediscovered by Valentine [17] using different methods. The writer [12] proved the same fact for a "monotone" function, and Grünbaum [9] combined these two theorems into one. A further improvement to the writer's theorem was given by Debrunner and Flor [6], who showed that the desired new functional value could always be chosen in the convex hull of the given functional values; several different proofs of this fact have now been given (see [14], [3]). An easy consequence of Kirszbraun's theorem is that a Lipschitz function in Hilbert space with maximal domain is everywhere-defined (see [11], [13]).

It was shown by S. Banach [1] that a real-valued function defined on a subset of a metric space and satisfying $|f(y_1)-f(y_2)| \leqq [\delta(y_1, y_2)]^\alpha$, with $0<\alpha\leqq 1$ (we call this "Lipschitz-Hölder continuity"), can be extended to the whole metric space so as to satisfy the same inequality. Banach's theorem was rediscovered by Czipszer and Gehér [4] in case $\alpha=1$ (but note that Banach's result follows, since $[\delta(y_1, y_2)]^\alpha$ is another metric if $\alpha\leqq 1$). For a general review of the above subjects, see the article of Danzer, Grünbaum, and Klee [5]; see also [7].

In this paper, we give a unified method for proving all the above results, and also new theorems, the most striking of which is the following generalization of the Kirszbraun and Banach theorems:

THEOREM 1. *Let H be a Hilbert space, M a metric space, $D\subset M$. Suppose $f: D\to H$ satisfies $\|f(y_1)-f(y_2)\| \leqq [\delta(y_1, y_2)]^\alpha$ $(0<\alpha\leqq 1)$. Then there exists an extension of f to all of M satisfying the same inequality, if either*

(i) $\alpha\leqq\frac{1}{2}$, *or*

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