BANACH SPACES OF LIPSCHITZ FUNCTIONS AND VECTOR-VALUED LIPSCHITZ FUNCTIONS[1]

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Introduction. Given a metric space (S, d), and a Banach space E, let $\mathrm{Lip}_E(S, d)$ denote the vector space of bounded functions $f: S\to E$ such that

$$\|f\|_d = \sup\{\|f(s) - f(t)\|d^{-1}(s, t) \mid s \neq t\}$$

is finite. Let $\|\cdot\|_\infty$ denote the sup-norm. Then $\mathrm{Lip}_E(S, d)$ endowed with $\|\cdot\| = \max(\|\cdot\|_\infty, \|\cdot\|_d)$ is a Banach space. $\mathrm{lip}_E(S, d)$ denotes the closed subspace of functions f such that

$$\lim_{d(s,t)\to 0} \|f(s) - f(t)\|d^{-1}(s, t) = 0.$$

When E is the set of real or complex numbers, we drop the subscript and write $\mathrm{Lip}(S, d)$ and $\mathrm{lip}(S, d)$.

In this paper we examine the Banach space properties of $\mathrm{lip}(S, d)$ and $\mathrm{Lip}(S, d)$ and extend some known results. Details and proofs of results presented here will appear elsewhere.

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1. Weak completeness and extreme points.

THEOREM 1.1. *Let (S, d) be any metric space. A sequence $\{f_n\}$ in $\mathrm{lip}(S, d)$ is weakly Cauchy if and only if it is bounded and every sequence $\{s_m\}$ in S has a subsequence $\{s_{m_i}\}$ such that* $\lim_{n\to\infty} \lim_{i\to\infty} f_n(s_{m_i})$ *exists.*

COROLLARY 1.2. *If (S, d) is compact, $\{f_n\}$ is weakly Cauchy if and only if it is bounded and* $\lim_{n\to\infty} f_n(s)$ *exists for each $s\in S$.*

If $0<\alpha\leq 1$ and d is a metric, so is d^α. We frequently consider $\mathrm{lip}(S, d^\alpha)$ for $0<\alpha<1$, since this space separates points.

THEOREM 1.3. *Let (S, d) be any metric space and $0<\alpha<1$. Then neither* $\mathrm{Lip}(S, d^\alpha)$ *nor* $\mathrm{lip}(S, d^\alpha)$ *is weakly sequentially complete unless S is a finite set.*

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